

TRIESTE LECTURES ON SOLITONS IN NONCOMMUTATIVE GAUGE THEORIES

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We present a pedagogical introduction into noncommutative gauge theories, their stringy origin, and non-perturbative effects, including monopole and instantons solutions.

1 Introduction

Recently there has been a revival of interest in noncommutative gauge theories^{86,14,21}. They are interesting examples of nonlocal field theories which in the certain limit (of large noncommutativity) become essentially equivalent to the large N ordinary gauge theories^{37,27}; certain supersymmetric versions of noncommutative gauge theories arise as $\alpha' \rightarrow 0$ limit of theories on Dp-branes in the presence of background B -field^{24,82}; the related theories arise in Matrix compactifications with C -field turned on¹⁵; finally, noncommutativity is in some sense an intrinsic feature of the open string field theory^{93,54,80}.

A lot of progress has been recently achieved in the analysis of the classical solutions of the noncommutative gauge theory on the noncommutative versions of Minkowski or Euclidean spaces. The first explicit solutions and their moduli were analyzed in⁷⁴ where instantons in the four dimensional noncommutative gauge theory (with self-dual noncommutativity) were constructed. These instantons play an important role in the construction of the discrete light cone quantization of the M-theory fivebrane^{3,2}, and they also gave a hope of giving an interpretation in the physical gauge theory language of the torsion free sheaves which appear in various interpretations of D-brane states^{59,43}, in particular those responsible for the entropy of black holes realized via D5-D1 systems⁸⁸, and also enter the S-duality invariant partition functions of $\mathcal{N} = 4$ super-Yang-Mills theory⁹². One can also relax the self-duality assumption on the noncommutativity. The construction of⁷⁴ easily generalizes⁷² to the general case $\theta \neq 0$ (see also^{31,32}). In addition to the instantons (which are particles in 4+1 dimensional theory), which represent the D0-D4 system, the monopole-like solutions were found³⁹ in U(1) gauge theory in 3+1 dimensions. The latter turn out to have a string attached to them. The string with the monopole at its end are the noncommutative field theory

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realization of the D3-D1 system, where D1 string ends on the D3 brane and bends at some specific angle towards the brane. One can also find the solutions describing the string itself^{77, 40}, both the BPS and in the non-BPS states; also the dimensionally reduced solutions in 2+1 dimensions^{40,1}, describing the D0-D2 systems; finite length strings, corresponding to $U(2)$ monopoles⁴¹.

The solitonic strings described in these lectures all carry magnetic fluxes. Their S-dual electric flux strings represent fundamental strings located nearby the D-branes with space-time noncommutativity. Also we do not consider theories on the noncommutative compact manifolds, like tori. Some classical and quantum aspects of the Yang-Mills theory on the noncommutative tori are analyzed in⁷⁸.

These lectures are organized as follows. The section 2 contains a pedagogical introduction into noncommutative gauge theories. Section 3 explains how noncommutative gauge theories arise as $\alpha' \rightarrow 0$ limits of open string theory. The section 4 constructs instantons in noncommutative gauge theory on \mathbf{R}^4 for any group $U(N)$. The section 5 presents explicit formulae for the $U(1)$ gauge group. The section 6 presents monopole solutions in $U(1)$ and $U(2)$ noncommutative gauge theories. The section 7 is devoted to some historic remarks. The format of these lectures does not allow to cover all interesting aspects of the noncommutative gauge theories, both classical and quantum, and their relations to string/M-theory, and to large N ordinary gauge theories. We refer the interested readers to the review²³ which will address all these issues in greater detail.

•Exercises are printed with the help of small fonts. The symbol \mathbf{i} denotes $\sqrt{-1}$, not to be confused with the space-time index i . The letter θ will be used only for the Poisson tensor θ^{ij} , or its components, while gauge theory theta angle will be denoted by ϑ . The symbol $*$ denotes Hodge star, while \star stands for star-products.

2 Noncommutative Geometry and Noncommutative Field Theory

2.1 A brief mathematical introduction

It has been widely appreciated by the mathematicians (starting with the seminal works of Gelfand, Grothendieck, and von Neumann) that the geometrical properties of a space X are encoded in the properties of the commutative algebra $C(X)$ of the continuous functions $f : X \rightarrow \mathbf{C}$ with the ordinary rules of point-wise addition and multiplication: $(f + g)(x) = f(x) + g(x)$, $f \cdot g(x) = f(x)g(x)$.

More precisely, $C(X)$ knows only about the topology of X , but one can

refine the definitions and look at the algebra $C^\infty(X)$ of the smooth functions or even at the DeRham complex $\Omega(X)$ to decipher the geometry of X .

The algebra $\mathcal{A} = C(X)$ is clearly associative, commutative and has a unit ($\mathbf{1}(x) = 1$). It also has an involution, which maps a function to its complex conjugate: $f^\dagger(x) = \overline{f(x)}$.

The points x of X can be viewed in two ways: as maximal ideals of \mathcal{A} : $f \in \mathcal{I}_x \Leftrightarrow f(x) = 0$; or as the irreducible (and therefore one-dimensional for \mathcal{A} is commutative) representations of \mathcal{A} : $R_x(f) = f(x)$, $R_x \approx \mathbf{C}$.

The vector bundles over X give rise to projective modules over \mathcal{A} . Given a bundle E let us consider the space $\mathcal{E} = \Gamma(E)$ of its sections. If $f \in \mathcal{A}$ and $\sigma \in \mathcal{E}$ then clearly $f\sigma \in \mathcal{E}$. This makes \mathcal{E} a representation of \mathcal{A} , i.e. a module. Not every module over \mathcal{A} arises in this way. The vector bundles over topological spaces have the following remarkable property, which is the content of Serre-Swan theorem: for every vector bundle E there exists another bundle E' such that the direct sum $E \oplus E'$ is a trivial bundle $X \times \mathbf{C}^N$ for sufficiently large N . When translated to the language of modules this property reads as follows: for the module \mathcal{E} over \mathcal{A} there exists another module \mathcal{E}' such that $\mathcal{E} \oplus \mathcal{E}' = F_N = \mathcal{A}^{\oplus N}$. We have denoted by $F_N = \mathcal{A} \otimes_{\mathbf{C}} \mathbf{C}^N$ the so-called free module over \mathcal{A} of rank N . Unless otherwise stated the symbol \otimes below will be used for tensor products over \mathbf{C} . The modules with this property are called *projective*. The reason for them to be called in such a way is that \mathcal{E} is an image of the free module F_N under the projection which is identity on \mathcal{E} and zero on \mathcal{E}' . In other words, for each projective module \mathcal{E} there exists N and an operator $P \in \text{Hom}(F_N, F_N)$, such that $P^2 = P$, and $\mathcal{E} = P \cdot F_N$.

Noncommutative geometry relaxes the condition that \mathcal{A} must be commutative, and develops a geometrical intuition about the noncommutative associative algebras with anti-holomorphic involution † (\mathbf{C}^* -algebras).

In particular, the notion of vector bundle over X is replaced by the notion of the projective module over \mathcal{A} . Now, when \mathcal{A} is noncommutative, there are two kinds of modules: left and right ones. The left \mathcal{A} -module is the vector space M_l with the operation of left multiplication by the elements of the algebra \mathcal{A} : for $m \in M_l$ and $a \in \mathcal{A}$ there must be an element $am \in M_l$, such that for a_1, a_2 : $a_1(a_2m) = (a_1a_2)m$. The definition of the right \mathcal{A} -module M_r is similar: for $m \in M_r$ and $a \in \mathcal{A}$ there must be an element $ma \in M_r$, such that for a_1, a_2 : $(ma_1)a_2 = m(a_1a_2)$. The free module $F_N = \mathcal{A} \oplus \dots_{N \text{ times}} \oplus \mathcal{A} = \mathcal{A} \otimes \mathbf{C}^N$ is both left and right one. The projective \mathcal{A} -modules are defined just as in the commutative case, except that for the left projective \mathcal{A} -module \mathcal{E} the module \mathcal{E}' , such that $\mathcal{E} \oplus \mathcal{E}' = F_N$, also must be left, and similarly for the right modules.

The manifolds can be mapped one to another by means of smooth maps: $g : X_1 \rightarrow X_2$. The algebras of smooth functions are mapped in the opposite

way: $g^* : C^\infty(X_2) \rightarrow C^\infty(X_1)$, $g^*(f)(x_1) = f(g(x_1))$. The induced map of the algebras is the algebra homomorphism:

$$g^*(f_1 f_2) = g^*(f_1)g^*(f_2), \quad g^*(f_1 + f_2) = g^*(f_1) + g^*(f_2)$$

Naturally, the smooth maps between two manifolds are replaced by the homomorphisms of the corresponding algebras. In particular, the maps of the manifold to itself form the associative algebra $Hom(\mathcal{A}, \mathcal{A})$. The diffeomorphisms would correspond to the invertible homomorphisms, i.e. automorphisms $Aut(\mathcal{A})$. Among those there are internal, generated by the invertible elements of the algebra:

$$a \mapsto g^{-1}ag$$

The infinitesimal diffeomorphisms of the ordinary manifolds are generated by the vector fields $V^i \partial_i$, which differentiate functions,

$$f \mapsto f + \varepsilon V^i \partial_i f$$

In the noncommutative setup the vector field is replaced by the derivation of the algebra $V \in Der(\mathcal{A})$:

$$a \mapsto a + \varepsilon V(a), \quad V(a) \in \mathcal{A}$$

and the condition that $V(a)$ generates an infinitesimal homomorphism reads as:

$$V(ab) = V(a)b + aV(b)$$

which is just the definition of the derivation. Among various derivations there are internal ones, generated by the elements of the algebra itself:

$$V_c(a) = [a, c] := ac - ca, \quad c \in \mathcal{A}$$

These infinitesimal diffeomorphisms are absent in the commutative setup, but they have close relatives in the case of Poisson manifold X .

2.2 Flat noncommutative space

The basic example of the noncommutative algebra which will be studied here is the enveloping algebra of the Heisenberg algebra. Consider the Euclidean space \mathbf{R}^d with coordinates x^i , $i = 1, \dots, d$. Suppose a constant antisymmetric matrix θ^{ij} is fixed. It defines a Poisson bi-vector field $\frac{1}{2}\theta^{ij}\partial_i \wedge \partial_j$ and therefore the noncommutative associative product on \mathbf{R}^d . The coordinate functions x^i on

the deformed noncommutative manifold will obey the following commutation relations:

$$[x^i, x^j] = \mathbf{i}\theta^{ij} , \quad (1)$$

We shall call the algebra \mathcal{A}_θ (over \mathbf{C}) generated by the x^i satisfying (1), together with convergence conditions on the allowed expressions of the x^i – the noncommutative space-time. The algebra \mathcal{A}_θ has an involution $a \mapsto a^\dagger$ which acts as a complex conjugation on the central elements $(\lambda \cdot \mathbf{1})^\dagger = \bar{\lambda} \cdot \mathbf{1}$, $\lambda \in \mathbf{C}$ and preserves x^i : $(x^i)^\dagger = x^i$. The elements of \mathcal{A}_θ can be identified with ordinary complex-valued functions on \mathbf{R}^d , with the product of two functions f and g given by the Moyal formula (or star product):

$$f \star g(x) = \exp \left[\frac{\mathbf{i}}{2} \theta^{ij} \frac{\partial}{\partial x_1^i} \frac{\partial}{\partial x_2^j} \right] f(x_1) g(x_2) \Big|_{x_1=x_2=x} . \quad (2)$$

Fock space formalism.

By an orthogonal change of coordinates we can map the Poisson tensor θ_{ij} onto its canonical form:

$$x^i \mapsto z_a, \bar{z}_a, \quad a = 1, \dots, r ; \quad y_b, \quad b = 1, \dots, d - 2r,$$

so that:

$$[y_a, y_b] = [y_b, z_a] = [y_b, \bar{z}_a] = 0, \quad [z_a, \bar{z}_b] = -2\theta_a \delta_{ab}, \quad \theta_a > 0 \quad (3)$$

$$ds^2 = dx_i^2 + dy_b^2 = dz_a d\bar{z}_a + dy_b^2.$$

Since $z(\bar{z})$ satisfy (up to a constant) the commutation relations of creation (annihilation) operators we can identify functions $f(x, y)$ with the functions of the y_a valued in the space of operators acting in the Fock space \mathcal{H}_r of r creation and annihilation operators:

$$\mathcal{H}_r = \bigoplus_{\vec{n}} \mathbf{C} |n_1, \dots, n_r\rangle \quad (4)$$

$$c_a = \frac{1}{\sqrt{2\theta_a}} \bar{z}_a, \quad c_a^\dagger = \frac{1}{\sqrt{2\theta_a}} z_a, \quad [c_a, c_b^\dagger] = \delta_{ab}$$

$$c_a |\vec{n}\rangle = \sqrt{n_a} |\vec{n} - 1_a\rangle, \quad c_a^\dagger |\vec{n}\rangle = \sqrt{n_a + 1} |\vec{n} + 1_a\rangle$$

Let $\hat{n}_a = c_a^\dagger c_a$ be the a 'th number operator.

The Hilbert space \mathcal{H}_r is the example of left projective module over the algebra \mathcal{A}_θ . Indeed, consider the element $P_0 = |\vec{0}\rangle\langle\vec{0}| \sim \exp - \sum_a \frac{z_a \bar{z}_a}{\theta_a}$. It obeys $P_0^2 = P_0$, i.e. it is a projector. Consider the rank one free module $F_1 = \mathcal{A}_\theta$ and let us consider its left sub-module, spanned by the elements of the form: $f \star P_0$. As a module it is clearly isomorphic to \mathcal{H}_r , isomorphism being: $|\vec{n}\rangle \mapsto |\vec{n}\rangle\langle\vec{0}|$. It is projective module, the complementary module being $\mathcal{A}_\theta(1 - P_0) \subset \mathcal{A}_\theta$.

The procedure that maps ordinary commutative functions onto operators in the Fock space acted on by z_a, \bar{z}_a is called Weyl ordering and is defined by:

$$f(x) \mapsto \hat{f}(z_a, \bar{z}_a) = \int f(x) \frac{d^{2r}x d^{2r}p}{(2\pi)^{2r}} e^{i(\bar{p}_a z_a + p_a \bar{z}_a - p \cdot x)}. \quad (5)$$

• Show that if $f \mapsto \hat{f}, g \mapsto \hat{g}$ then $f \star g \mapsto \hat{f}\hat{g}$.

Integration over the noncommutative space

Weyl ordering also allows to express the integrals of the ordinary functions over the commutative space in terms of the traces of the operators, corresponding to them:

$$\int d^{2r}x f(x) = (2\pi\theta)^r \text{Tr}_{\mathcal{H}_r} \hat{f}, \quad (6)$$

as follows immediately from (5). Sometimes the integral reduces to the boundary term. What is the boundary term in the noncommutative, Fock space setup?

To be specific, let us consider the case $r = 1$. The general case follows trivially. Consider the integral

$$\int d^2x (\partial_1 f_1 + \partial_2 f_2) = (2\pi i) \text{Tr}_{\mathcal{H}} \left([\hat{f}_1, x^2] + [x^1, \hat{f}_2] \right) = \pi\sqrt{2\theta} \text{Tr} ([c, \mathbf{f}] - [c^\dagger, \mathbf{f}^\dagger]) \quad (7)$$

$\mathbf{f} = \hat{f}_1 + i\hat{f}_2$. In computing the trace

$$\text{Tr}_{\mathcal{H}} [c, \mathbf{f}] = \sum_n \langle n | [c, \mathbf{f}] | n \rangle, \quad (8)$$

we get naively zero, for the trace of a commutator usually vanishes. But we should be careful, since the matrices are infinite and the trace is an infinite sum. If we regulate it by restricting the sum to $n \leq N$, then the matrix element $\langle N | c | N+1 \rangle \langle N+1 | \mathbf{f} | N \rangle$ is not cancelled, so that the regularized trace is

$$\text{Tr}_{\mathcal{H}_N} [c, \mathbf{f}] = \sqrt{N+1} \langle N+1 | \mathbf{f} | N \rangle \quad (9)$$

and similarly for c^\dagger . Thus:

$$\oint_{\infty} f_1 dx^2 - f_2 dx^1 = 2\pi \sqrt{2\theta(N+1)} \text{Re} \langle N+1 | \hat{f}_1 + i\hat{f}_2 | N \rangle_{N \rightarrow \infty} \quad (10)$$

• Consider $\mathbf{f} = \frac{1}{c^\dagger c} c^\dagger$. Compute the integral (7) directly and via (10).

Let us conclude this section with the remark that (6) defines the trace on a part of the algebra \mathcal{A}_θ consisting of the trace class operators. They correspond to the space $L^1(\mathbf{R}^{2r})$ of integrable functions. Even though the trace is taken in the specific representation of \mathcal{A}_θ it is defined intrinsically, thanks to (6). In what follows we omit the subscript \mathcal{H}_r in the notation for the trace on \mathcal{A}_θ .

Symmetries of the flat noncommutative space

The algebra (1) has an obvious symmetry: $x^i \mapsto x^i + \varepsilon^i$, with $\varepsilon^i \in \mathbf{R}$. For invertible Poisson structure θ , such that $\theta_{ij}\theta^{jk} = \delta_i^k$ this symmetry is an example of the internal automorphism of the algebra:

$$a \mapsto e^{i\theta_{ij}\varepsilon^i x^j} a e^{-i\theta_{ij}\varepsilon^i x^j} \quad (11)$$

In addition, there are rotational symmetries which we shall not need.

2.3 Gauge theory on noncommutative space

In an ordinary gauge theory with gauge group G the gauge fields are connections in some principal G -bundle. The matter fields are the sections of the vector bundles with the structure group G . Sections of the noncommutative vector bundles are elements of the projective modules over the algebra \mathcal{A}_θ .

Gauge fields, matter fields

In the ordinary gauge theory the gauge field arises through the operation of covariant differentiation of the sections of a vector bundle. In the noncommutative setup the situation is similar. Suppose M is a projective module over \mathcal{A}_θ . The connection ∇ is the operator

$$\nabla : \mathbf{R}^d \times M \rightarrow M, \quad \nabla_\varepsilon(m) \in M, \quad \varepsilon \in \mathbf{R}^d, m \in M,$$

where \mathbf{R}^d denotes the commutative vector space, the Lie algebra of the automorphism group generated by (11). The connection is required to obey the Leibnitz rule:

$$\nabla_\varepsilon(am_1) = \varepsilon^i(\partial_i a)m_1 + a\nabla_\varepsilon m_1 \quad (12)$$

$$\nabla_\varepsilon(m_{\mathbf{r}}a) = m_{\mathbf{r}}\varepsilon^i(\partial_i a) + (\nabla_\varepsilon m_{\mathbf{r}})a . \quad (13)$$

Here, (12) is the condition for left modules, and (13) is the condition for the right modules. As usual, one defines the curvature $F_{ij} = [\nabla_i, \nabla_j]$ - the operator $\Lambda^2 \mathbf{R}^d \times M \rightarrow M$ which commutes with the multiplication by $a \in \mathcal{A}_\theta$. In other words, $F_{ij} \in \text{End}_{\mathcal{A}}(M)$. In ordinary gauge theories the gauge fields come with gauge transformations. In the noncommutative case the gauge transformations, just like the gauge fields, depend on the module they act in. For the module M the group of gauge transformations \mathcal{G}_M consists of the invertible endomorphisms of M which commute with the action of \mathcal{A} on M :

$$\mathcal{G}_M = \text{GL}_{\mathcal{A}_\theta}(M)$$

Its Lie algebra $\text{End}_{\mathcal{A}}(M)$ will also be important for us.

All the discussion above can be specified to the case where the module has a Hermitian inner product, with values in \mathcal{A}_θ .

In addition to gauge fields gauge theories often have matter fields. One should distinguish two types of matter fields. First of all, the elements φ of the module M where ∇ acts, can be used as the matter fields. Then $\nabla_i \varphi$ is the usual covariant derivative of the element of the module. This is the non-commutative analogue of the matter fields in the fundamental representation of the gauge group. Another possibility is to look at the Lie algebra of the gauge group $\text{End}_{\mathcal{A}_\theta}(M)$. Its elements Φ (we shall loosely call them adjoint Higgs fields) commute with the action of the algebra \mathcal{A} in the module, but act nontrivially on the elements of the module M : $\varphi \mapsto \Phi \cdot \varphi \in M$. In particular, one can consider the commutators between the covariant derivatives and the Higgs fields: $[\nabla_i, \Phi] \in \text{End}_{\mathcal{A}_\theta}(M)$. These commutators can be called the covariant derivatives of the adjoint Higgs fields.

The important source of such matter fields is the dimensional reduction of the higher dimensional theory. Then the components of the covariant derivative operator in the collapse directions become the adjoint Higgs fields in the reduced theory.

Fock module and connections there.

Recall that the algebra \mathcal{A}_θ for $d = 2r$ and non-degenerate θ has an important irreducible representation, the left module \mathcal{H}_r . Let us now ask, what kind of connections does the module \mathcal{H}_r have?

By definition(12), we are looking for a collection of operators $\nabla_i : \mathcal{H}_r \rightarrow \mathcal{H}_r$, $i = 1, \dots, 2r$, such that:

$$[\nabla_i, a] = \partial_i a$$

for any $a \in \mathcal{A}$. Using the fact that $\partial_i a = i\theta_{ij}[x^j, a]$ and the irreducibility of \mathcal{H}_r we conclude that:

$$\nabla_i = i\theta_{ij}x^j + \kappa_i, \quad \kappa_i \in \mathbf{C} \quad (14)$$

If we insist on unitarity of ∇ , then $i\kappa_i \in \mathbf{R}$. Thus, the space of all gauge fields suitable for acting in the Fock module is rather thin, and is isomorphic to the vector space \mathbf{R}^d (which is canonically dual to the Lie algebra of the automorphisms of \mathcal{A}_θ). The gauge group for the Fock module, again due to its irreducibility is simply the group $U(1)$, which multiplies all the vectors in \mathcal{G}_r by a single phase. In particular, it preserves κ_i 's, so they are gauge invariant. It remains to find out what is the curvature of the gauge field given by (14). The straightforward computation of the commutators gives:

$$F_{ij} = i\theta_{ij} \quad (15)$$

i.e. all connections in the Fock module have the constant curvature.

Free modules and connections there.

If the right (left) module M is free, i.e. it is a sum of several copies of the algebra \mathcal{A}_θ itself, then the connection ∇_i can be written as

$$\nabla_i = \partial_i + A_i$$

where A_i is the operator of the left (right) multiplication by the matrix with \mathcal{A}_θ -valued entries:

$$\nabla_i m_l = \partial_i m_l + m_l A_i, \quad \nabla_i m_r = \partial_i m_r + A_i m_r \quad (16)$$

In the same operator sense the curvature obeys the standard identity:

$$F_{ij} = \partial_i A_j - \partial_j A_i + A_i A_j - A_j A_i .$$

Given a module M over some algebra \mathcal{A} one can multiply it by a free module $\mathcal{A}^{\oplus N}$ to make it a module over an algebra $\text{Mat}_{N \times N}(\mathcal{A})$ of matrices with elements from \mathcal{A} . In the non-abelian gauge theory over \mathcal{A} we are interested in projective modules over $\text{Mat}_{N \times N}(\mathcal{A})$. If the algebra \mathcal{A} (or perhaps its subalgebra) has a trace, Tr , then the algebra $\text{Mat}_{N \times N}(\mathcal{A})$ has a trace given by the composition of a usual matrix trace and Tr .

It is a peculiar property of the noncommutative algebras that the algebras \mathcal{A} and $\text{Mat}_{N \times N}(\mathcal{A})$ have much in common. These algebras are called Morita equivalent and under some additional conditions the gauge theories over \mathcal{A} and over $\text{Mat}_{N \times N}(\mathcal{A})$ are also equivalent. This phenomenon is responsible for the similarity between the "abelian noncommutative" and "non-abelian commutative" theories.

Observables from gauge fields

Just like in the commutative case the difference of two connections in the module M is the operator from $\text{End}_{\mathcal{A}_\theta}(M)$. Thus we can write:

$$\nabla_i = \mathbf{i}\theta_{ij}x^j + D_i, \quad D_i \in \text{End}_{\mathcal{A}_\theta}(M) \quad (17)$$

The curvature of the connection ∇_i is, therefore:

$$F_{ij} = [\nabla_i, \nabla_j] = \mathbf{i}\theta_{ij} + [D_i, D_j] \in \text{End}_{\mathcal{A}_\theta}(M) \quad (18)$$

For free modules the operators D_i appear from the formulae (16) if we represent ∂_i as $\mathbf{i}\theta_{ij}[x^j, \cdot]$:

$$\nabla_i m_{\mathbf{l}} = \mathbf{i}\theta_{ij}x^j m_{\mathbf{r}} + m_{\mathbf{r}} D_i, \quad \nabla_i m_{\mathbf{r}} = -m_{\mathbf{r}} \mathbf{i}\theta_{ij}x^j + D_i^\dagger m_{\mathbf{r}} \quad (19)$$

The relation of the operators D_i and the conventional gauge fields A_i is

$$D_i = -\mathbf{i}\theta_{ij}x^j + A_i \quad (20)$$

Going back to the generic case, we shall now describe a (overcomplete) set of gauge invariant observables in the gauge theory on the module M . The gauge transformations act on the operators D_i “locally”, i.e. for $g \in \text{GL}_{\mathcal{A}_\theta}(M)$:

$$D_i \mapsto g^{-1} D_i g \quad (21)$$

Hence the spectrum of the operators D_i , or of any analytic function of them is gauge invariant. In particular, the following observables are the noncommutative analogues of the Wilson loops in the ordinary gauge theory. Choose a contour on the ordinary space \mathbf{R}^{2r} : $\gamma^i(t)$, $0 \leq t \leq 1$. Define an operator

$$\mathcal{U}_\gamma = P \exp \int_0^1 dt D_i \dot{\gamma}^i(t) \quad (22)$$

which also transforms as in (21). The following observable

$$\mathcal{W}_\gamma = \text{Tr}_M \mathcal{U}_\gamma \quad (23)$$

is gauge-invariant. It is typically a distribution on the space of contours γ . For example, for the Fock module

$$\mathcal{W}_\gamma = e^{\kappa_i(\gamma^i(1) - \gamma^i(0))} \times \text{Tr}_{\mathcal{H}_r} 1$$

while for the free module in the vacuum ($D_i = -\mathbf{i}\theta_{ij}x^j$):

$$\mathcal{W}_\gamma = \delta^{2r} (\theta_{ij} (\gamma^j(1) - \gamma^j(0))) \exp \mathbf{i} \left[\oint_\gamma \theta_{ij} \gamma^i d\gamma^j \right]$$

The operators (23) are closely related to the “noncommutative momentum carrying Wilson loops” considered in⁵⁰.

2.4 Lagrangian, and couplings

In order to write down the Lagrangian for the gauge theory in the commutative setup one needs to specify a few details about the space-time and the gauge theory: the space-time metric $G_{\mu\nu}$, gauge coupling g_{YM} , theta angle ϑ and so on. The same is true for the noncommutative theory, except that the parameters above are more restricted, for given \mathcal{A} . In these lectures we shall be dealing with static field configurations, in the theories in $p+1$ dimensions. We shall only look at the potential energy for such configurations. It is given by:

$$\mathcal{E}(A) = -\frac{1}{4g_{\text{YM}}^2} \sum \sqrt{G} G^{ii'} G^{jj'} \text{Tr} F_{ij} F_{i'j'} . \quad (24)$$

If additional adjoint Higgs matter fields Φ are present then the (24) becomes:

$$\mathcal{E}(A, \Phi) = \mathcal{E}(A) + \sum \sqrt{G} G^{ii'} \text{Tr} \nabla_i \Phi \nabla_{i'} \Phi + \dots \quad (25)$$

These formulae make sense for the constant Euclidean metric G_{ij} only. String theory allows more general backgrounds, where the closed string metric g and the B field both are allowed to be non-constant. They are presumably described by more abstract techniques of noncommutative geometry⁴ which we shall not use in these lectures.

We now proceed with the exposition of how the associative algebras, their deformations, and gauge theories over them arise in the string theory.

3 Noncommutative geometry and strings in background B -fields

3.1 Conventional strings and D -branes

Let us look at the theory of open strings in the following closed string background: Flat space X , metric g_{ij} , constant Neveu-Schwarz B -field B_{ij} ; Dp -branes are present, so that $B_{ij} \neq 0$ for some i, j along the branes. The presence of the Dp -branes means that the B_{ij} cannot be gauged away - if we try to get rid of B_{ij} by means of the gauge transformation

$$B \rightarrow B + d\Lambda$$

then we create a gauge field A_i on the brane, whose field strength F_{ij} is exactly equal to B_{ij} .

The bosonic part of the worldsheet action of our string is given by:

$$S = \frac{1}{4\pi\alpha'} \int_{\Sigma} (g_{ij} \partial_a x^i \partial^a x^j - 2\pi i \alpha' B_{ij} \epsilon^{ab} \partial_a x^i \partial_b x^j) \quad (26)$$

$$\frac{1}{4\pi\alpha'} \int_{\Sigma} g_{ij} \partial_a x^i \partial^a x^j - \frac{i}{2} \int_{\partial\Sigma} B_{ij} x^i \partial_t x^j$$

Here we denote by ∂_t the tangential derivative along the boundary $\partial\Sigma$ of the worldsheet Σ (which we shall momentarily take to be the upper half-plane). We shall also need the normal derivative ∂_n .

The boundary conditions which follow from varying the action (26) are^b:

$$g_{ij} \partial_n x^i + 2\pi i \alpha' B_{ij} \partial_t x^j|_{\partial\Sigma} = 0 \quad (27)$$

Now, on the upper half-plane with the coordinate $z = t + iy$, $y > 0$ we can compute the propagator:

$$\langle x^i(z) x^j(w) \rangle = \quad (28)$$

$$-\alpha' \left[g^{ij} \log \left(\frac{z-w}{z-\bar{w}} \right) + G^{ij} \log |z-w|^2 + \frac{1}{2\pi\alpha'} \theta^{ij} \log \left(\frac{z-\bar{w}}{\bar{z}-w} \right) + D^{ij} \right]$$

where D^{ij} is independent of z, w ,

$$G^{ij} = \left(\frac{1}{g + 2\pi\alpha' B} \right)_S^{ij} \quad (29)$$

$$\theta^{ij} = 2\pi\alpha' \left(\frac{1}{g + 2\pi\alpha' B} \right)_A^{ij}$$

where S and A denote the symmetric and antisymmetric parts respectively. The open string vertex operators are given by the expressions like

$$: f(x(t), \partial_t x(t), \partial_t^2 x(t), \dots) : \quad (30)$$

where everything is evaluated at $y = 0$.

The properties of the open strings are encoded in the operator product expansion of the open string vertex operators. By specifying (28) at $y = 0$ we get

^bIt was observed by S. Shatashvili in 1995 right after the appearance of \tilde{r}^6 that the B -field (or equivalently the constant electromagnetic field) as well as the tachyon interpolates between the Dirichlet and von Neumann boundary conditions, and as a consequence between different solutions of the target space lagrangian of background independent open string field theory^{94,85}, and that by taking some of the components of $B_{ij} \rightarrow \infty$ one creates lower dimensional D-branes. Similar properties of the tachyon backgrounds are now extensively investigated^{34,56,35,7,83,89}

the expression for the propagator of the boundary values of the coordinates $x^j(t)$:

$$\langle x^i(t)x^j(s) \rangle = -\alpha' G^{ij} \log(t-s)^2 + \frac{\mathbf{i}}{2} \theta^{ij} \epsilon(t-s) \quad (31)$$

where $\epsilon(t) = -1, 0, +1$ for $t < 0, t = 0, t > 0$ respectively. From this expression we deduce :

$$[x^i, x^j] := T(x^i(t-0)x^j(t) - x^i(t+0)x^j(t)) = i\theta^{ij} \quad (32)$$

It means that the end-points of the open strings live on the noncommutative space where:

$$[x^i, x^j] = \mathbf{i}\theta^{ij}$$

with θ^{ij} being a constant antisymmetric matrix. Similarly, the OPE of the vertex operators:

$$V_p(t) =: e^{\mathbf{i}p \cdot x} : (t)$$

is given by :

$$V_p(t)V_q(s) = (t-s)^{2\alpha' G^{ij} p_i q_j} e^{-\frac{\mathbf{i}}{2} \theta^{ij} p_i q_j} V_{p+q}(s) \quad (33)$$

Seiberg and Witten suggested⁸² to consider the limit $\alpha' \rightarrow 0$ with G, θ being kept fixed. In this limit the OPE (33) goes over to the formula for the modified multiplication law on the ordinary functions on a space with the coordinates x :

$$V_p V_q = e^{-\frac{\mathbf{i}}{2} \theta^{ij} p_i q_j} V_{p+q} \quad (34)$$

(the appearance of the noncommutative algebra (34) in the case of compactification on a shrinking torus was observed earlier by M. Douglas and C. Hull in²⁴). The algebra defined in (34) is isomorphic to Moyal algebra (2). The product (34) is associative but clearly noncommutative.

Witten⁹⁶ remarked that the $\alpha' \rightarrow 0$ limit with fixed g_{ij}, θ^{ij} makes the algebra of open string vertex operators (which is associative as the vertex algebra) to factorize into the product of the associative algebra of the string zero modes and the algebra of string oscillators (excited modes). This allows to see some stringy effects already at the level of the noncommutative field theories.

3.2 Effective action

Vertex operators (30) give rise to the space-time fields Φ_k propagating along the worldvolume of the Dp-brane. Their effective Lagrangian is obtained by

evaluating the disc amplitudes with (30) inserted at the boundary of the disc:

$$\int d^{p+1}x \sqrt{\det G} \text{Tr} \partial^{n_1} \Phi_1 \partial^{n_2} \Phi_2 \dots \partial^{n_k} \Phi_k \sim \quad (35)$$

$$\langle \prod_{m=1}^k : \mathbf{P}_m (\partial x(t_m), \partial^2 x(t_m), \dots) e^{ip_m \cdot x(t_m)} : \rangle$$

If we compare the effective actions of the theory at $\theta = 0$ and $\theta \neq 0$ the difference is very simple to evaluate: one should only take into account the extra phase factors from (33):

$$\langle \prod_{m=1}^k : \mathbf{P}_m (\partial x(t_m), \partial^2 x(t_m), \dots) e^{ip_m \cdot x(t_m)} : \rangle_{G, \theta} = \quad (36)$$

$$e^{-\frac{i}{2} \sum_{n>m} p_i^n \theta^{ij} p_j^m \epsilon(t_n - t_m)} \langle \prod_{m=1}^k : \mathbf{P}_m (\partial x(t_m), \partial^2 x(t_m), \dots) e^{ip_m \cdot x(t_m)} : \rangle_{G, 0}$$

(the exponent in (36) is depends only on the cyclic order of the operators, due to the antisymmetry of θ and the momentum conservation:

$$\sum_m p^m = 0$$

Using (34),(2) we can easily conclude that the terms like (35) go over to the terms like:

$$\int d^{p+1}x \sqrt{\det G} \text{Tr} \partial^{n_1} \Phi_1 \star \partial^{n_2} \Phi_2 \star \dots \star \partial^{n_k} \Phi_k \quad (37)$$

This result holds even without taking the Seiberg-Witten limit.

In general, the relation between the off-shell string field theory action and the boundary operators correlation functions in the worldsheet conformal theory depends on the type of string theory we are talking about. In the case of bosonic string in the framework of Witten-Shatashvili background independent open string field theory this relation reads as^{94,85}:

$$\mathcal{S} = \mathcal{Z} - \beta^i \frac{\partial}{\partial t^i} \mathcal{Z} \quad (38)$$

where $\mathcal{Z}(t)$ is the generating function of the boundary correlators:

$$\mathcal{Z}(t) = \langle \exp \oint_{\partial \Sigma} t^i \mathcal{O}_i \rangle$$

with \mathcal{O}_i running through some basis in the space of the open string vertex operators, t^i being the corresponding couplings, and β^i the β -function of the coupling t^i :

$$\beta^i = \Lambda \frac{d}{d\Lambda} t^i$$

The relation(38) derived in⁸⁵ is very restrictive, for β^i vanish exactly where $d\mathcal{S}$ does, and allows to calculate the exact expression for \mathcal{S} up to two space-time derivatives^{34,56}. For the superstring it seems⁵⁷ that $\mathcal{S} = \mathcal{Z}$. This approach was successfully applied to the study of D-branes with the B-field turned on^{18,75,7}.

3.3 Gauge theory from string theory

Open strings carry gauge fields. This follows from the presence in the spectrum of the allowed vertex operators of the operator of the following simple form:

$$e_i(p) : \partial_t x^i e^{ip \cdot x} : \longleftrightarrow A = A_i(x) dx^i \quad (39)$$

It has (classical) dimension 1 and deforms the worldsheet conformal field theory as follows:

$$S = S_0 - \mathbf{i} \oint_{\partial\Sigma} A_i(x) \partial_t x^i dt \quad (40)$$

This deformation has the following naive symmetry:

$$\delta A_j = \partial_j \varepsilon \quad (41)$$

$\varepsilon(x)$ is the tachyonic vertex operator. Let us see, whether (41) is indeed a symmetry. To this end we estimate the δ -variation of the disc correlation function:

$$\delta \int Dx e^{-S} = \langle i \oint \partial_t \varepsilon dt \rangle_S = \langle i \oint \partial_t \varepsilon dt \rangle_{S_0} \quad (42)$$

$$- \langle \oint \oint dt ds A_j(x(t)) \partial_t x^j(t) \partial_s \varepsilon(s) \rangle_{S_0} + \dots$$

Let us regularize (42) by point splitting, i.e. by understanding the s, t integration with $s \neq t$ (of course, there are other regularizations, their equivalence leads to important predictions concerning the noncommutative gauge theory, see⁸²). Then the total s -derivative needs not to decouple. Instead, it gives:

$$\begin{aligned} & - \langle \oint dt \int_{s \neq t} ds A_j(x(t)) \partial_t x^j(t) \partial_s \varepsilon(x(s)) \rangle_{S_0} = \\ & \langle \oint dt A_j(x(t)) \partial_t x^j(t) (\varepsilon(x(t+0)) - \varepsilon(x(t-0))) \rangle_{S_0} = \end{aligned}$$

$$= \langle \oint \varepsilon \star A - A \star \varepsilon \rangle_{S_0}$$

This calculation shows that the naive transformation (41) must be supplemented by the correction term:

$$\delta A_j = \partial_j \varepsilon + A_j \star \varepsilon - \varepsilon \star A_j \quad (43)$$

The formula (43) is exactly the gauge transformation of the gauge field in the gauge theory on the noncommutative space. Similarly, the effective action for the gauge fields in the presence of $\theta \neq 0$ becomes that of the Yang-Mills theory on the noncommutative space plus the corrections which vanish in the $\alpha' \rightarrow 0$ limit. In what follows we shall need the relation between various string theory moduli in the case of D3-branes in the background of the constant B -field:

Open and closed string moduli.

We want to consider the decoupling limit of a D3-brane in the Type IIB string theory in a background with a constant Neveu-Schwarz B -field. Let us recall the relation of the parameters of the actions (24), (25) and the string theory parameters, before taking the Seiberg-Witten limit⁸².

We start with the D3-brane whose worldvolume is occupying the 0123 directions, and turn on a B -field:

$$\frac{1}{2} B dx^1 \wedge dx^2 \quad (44)$$

The indices i, j below will run from 1 to 3. We assume that the closed string metric g_{ij} is flat, and the closed string coupling g_s is small. According to⁸² the gauge theory on the D3-brane is described by a Lagrangian, which, when restricted to time-independent fields, equals (25) with the parameters

$$G_{ij}, \theta^{ij}, g_{\text{YM}}^2,$$

which are related to

$$g_{ij}, B_{ij}, g_s$$

via (29) as follows:

$$G_{ij} = g_{ij} - (2\pi\alpha')^2 (B g^{-1} B)_{ij}, \quad \theta^{ij} = -(2\pi\alpha')^2 \left(\frac{1}{g + 2\pi\alpha' B} B \frac{1}{g - 2\pi\alpha' B} \right)^{ij} \quad (45)$$

$$g_{\text{YM}}^2 = 4\pi^2 g_s (\det(1 + 2\pi\alpha' g^{-1} B))^{\frac{1}{2}}.$$

Suppose the open string metric is Euclidean: $G_{ij} = \delta_{ij}$, then (44),(45) imply:

$$g = dx_3^2 + \frac{(2\pi\alpha')^2}{(2\pi\alpha')^2 + \theta^2} (dx_1^2 + dx_2^2), \quad B = \frac{\theta}{(2\pi\alpha')^2 + \theta^2}, \quad (46)$$

and

$$g_s = \frac{g_{\text{YM}}^2}{2\pi} \frac{\alpha'}{\sqrt{(2\pi\alpha')^2 + \theta^2}}. \quad (47)$$

The Seiberg-Witten limit is achieved by taking $\alpha' \rightarrow 0$ with $G, \theta, g_{\text{YM}}^2$ kept fixed. In this limit the effective action of the D3-brane theory will become that of the (super)Yang-Mills theory on a noncommutative space \mathcal{A}_θ . The relevant part of the energy functional is:

$$\mathcal{E} = \frac{1}{2g_{\text{YM}}^2} \int \text{Tr} (B_i \star B_i + \nabla_i \Phi \star \nabla_i \Phi), \quad (48)$$

where

$$B_i = \frac{i}{2} \varepsilon_{ijk} F_{jk}. \quad (49)$$

The fluctuations of the D3-brane in some distinguished transverse direction are described by the dynamics of the Higgs field. Of course the true D3-brane theory will have six adjoint Higgs fields, one per transverse direction. We are looking at one of them. Also notice that the expression for the energy (48) goes over to the case of N D3-branes, one simply replaces \mathcal{A}_θ by $\text{Mat}_N(\mathcal{A}_\theta)$.

3.4 Noncommutative geometry from topological string theory

String theory can be well-defined even in the presence of non-constant B -field. The absolute minima of the B -field energy are achieved on the flat B -fields, i.e. if $dB = 0$. In this case, if in addition the tensor B_{ij} is invertible one can define the inverse tensor $\theta^{ij} = (B^{-1})^{ij}$ which will obey Jacobi identity $\theta\partial\theta = 0$. It is not widely appreciated in the string theory literature that a string can actually propagate in the background of the non necessarily invertible Poisson bi-vector θ^{ij} (we leave the question of whether the Poisson property can also be relaxed to future investigations).

We shall now describe a version of a string theory, which is perfectly sensible even for the non-constant Poisson bi-vector field θ^{ij} , not necessarily invertible. To do this we shall start with the bosonic string action with the target space X and then take the $\alpha' \rightarrow 0$ limit *à la* Seiberg and Witten. For simplicity we make an analytic continuation $\theta^{ij} \rightarrow i\theta^{ij}$ (cf.^{54,13}).

Take the action (26) not assuming that g_{ij}, B_{ij} are constant and rewrite it in the first order form:

$$S = \frac{1}{4\pi\alpha'} \int_{\Sigma} (g_{ij} \partial_a x^i \partial^a x^j - 2\pi i \alpha' B_{ij} \epsilon^{ab} \partial_a x^i \partial_b x^j) \leftrightarrow \quad (50)$$

$$\int p_i \wedge dx^i - \pi\alpha' G^{ij} p_i \wedge *p_j - \frac{1}{2} \theta^{ij} p_i \wedge p_j$$

where

$$2\pi\alpha' G + \theta = 2\pi\alpha' (g + 2\pi\alpha' B)^{-1}$$

Now suppose we take $\alpha' \rightarrow 0$ limit keeping θ and G fixed. The remaining part $\int p \wedge dx + \frac{1}{2} \theta p \wedge p$ of the action (50) immediately exhibits an enhancement of the (gauge) symmetry¹³:

$$p_i \mapsto p_i - d\lambda_i - \partial_i \theta^{jk} p_j \lambda_k, \quad x^i \mapsto x^i + \theta^{ij} \lambda_j \quad (51)$$

This symmetry must be gauge fixed, at the cost of introduction of a sequence of ghosts, anti-ghosts, Lagrange multipliers and gauge conditions¹³. It is not the goal of these lectures to do so in full generality with regards to various types of target space diffeomorphisms invariance one might want to keep track of⁸. Let us just mention that the result of this gauge fixing procedure is the topological string theory, which has some similarities both with the type A and type B sigma models. The field content of this sigma model is the following. It is convenient to think of p_i and x^i as of the twisted super-fields, which simply mean that both p_i and x^i are differential forms on Σ having components of all degrees, 0, 1, 2. The original Lagrangian had only 0-th component of x^i and 1st component of p_i . In addition, there are auxiliary fields χ^i, H^i , which are zero forms of opposite statistics - χ of the fermionic while H of the bosonic one. The gauge fixing conditions restrict the components of the superfields p and x to be:

$$x_{(1)}^i = \star d\chi^i, \quad p_i^{(2)} = 0 \quad (52)$$

We are discussing an $\alpha' \rightarrow 0$ limit of an open string theory. It means that the worldsheet Σ has a boundary, $\partial\Sigma$. The fields x, p, χ, H obey certain boundary conditions: x obeys Neumann boundary conditions, H vanishes at $\partial\Sigma$, i.e. it obeys Dirichlet boundary conditions. In this sense the theory we are studying is that of a D-brane wrapping the zero section ($H = 0$) X of the tangent bundle TX to the target space X . The rest of boundary conditions is summarized in the formula for the propagator below. The field χ^i is constant at the boundary (fermionic Dirichlet condition).

The further treatment of the resulting theory is done in the perturbation series expansion⁵⁴ in θ around the classical solution $x^i(z) = x^i = \text{const.}$ We shall work on Σ being the upper half-plane $\text{Im}z > 0$. The propagator is most conveniently written in the superfield language:

$$\langle p_i(z)x^j(w) \rangle = \delta_i^j d\phi(z, w), \quad \phi(z, w) = \frac{1}{2\mathbf{i}} \log \frac{(z-w)(z-\bar{w})}{(\bar{z}-\bar{w})(\bar{z}-w)} \quad (53)$$

where d is the deRham differential on $\Sigma \times \Sigma$.

Bulk observables

The theory can be deformed by adding observables to the action. Any polyvector field $\alpha^{i_1 \dots i_q} \frac{\partial}{\partial x^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{i_q}} \in \Lambda^q TX$ on X defines an observable in the theory:

$$\mathcal{O}_\alpha = \alpha^{i_1 \dots i_q} p_{i_1} \dots p_{i_q} \quad (54)$$

which is again an inhomogeneous form on Σ . Its degree 2 component can be added to the action. In particular, the θ -term in the original action corresponds to the bi-vector $\frac{1}{2}\theta^{ij}\partial_i \wedge \partial_j$. If we write the action (before the gauge fixing) in the form:

$$\int p_i \wedge dx^i + \mathcal{O}_\alpha$$

where α is a generic polyvector field on X then the only condition is that $[\alpha, \alpha] = 0$ where $[\cdot, \cdot]$ is the Schoutens bracket on the polyvector fields:

$$[\alpha, \beta] = \sum_i \frac{\partial \alpha}{\partial x^i} \wedge \frac{\partial \beta}{\partial (\partial_i)} \pm (\alpha \leftrightarrow \beta) \quad (55)$$

(the sign is determined by the parity of the degrees of α, β , in such a way that $[\cdot, \cdot]$ defines a super-Lie algebra).

Boundary observables

The correlation functions of interest are obtained by inserting yet another observables at the boundary of Σ . These correspond to the differential forms on X :

$$\omega_{i_1 \dots i_q} dx^{i_1} \wedge \dots \wedge dx^{i_q} \mapsto \omega_{i_1 \dots i_q} \chi^{i_1} \dots \chi^{i_q}$$

Actually, one is interested in computing string amplitudes. It means that the positions of the vertex operators at the boundary must be integrated over, up to the $SL_2(\mathbf{R})$ invariance (the gauge fixed action still has (super)conformal invariance).

In particular, the three-point function on the disc (or two-point function on the upper-half-plane, as a function of the boundary condition at infinity, $x(\infty) = x$), produces the deformed product on the functions:

$$\langle f(x(0))g(x(1)) [h(x)\chi \dots \chi] (\infty) \rangle_{\theta, \Sigma=\text{disc}} = \int_X f \star g h \quad (56)$$

for $f, g \in \text{Fun}(X)$, $h \in \Omega^{\dim X}(X)$.

The \star -product defined by (56) turns out to be associative: $f \star (g \star l) = (f \star g) \star l$. This is a consequence of a more general set of Ward identities obeyed by the string amplitudes in the theory. For their description see ^{54, 13}. The perturbative expression for the \star -product following from (56) turns out to be non-covariant with respect to the changes of local coordinates in X . This is a consequence of a certain anomaly in the theory. It will be discussed in detail in ⁸.

From now on we go back to the case where X is flat and θ is constant.

4 Instantons in noncommutative gauge theories

We would like to study the non-perturbative objects in noncommutative gauge theory.

In this lecture specifically we shall be interested in four dimensional instantons. They either appear as instantons themselves in the Euclidean version of the four dimensional theory (theory on Euclidean D3-brane), as solitonic particles in the theory on D4-brane, i.e. in 4+1 dimensions, or as instanton strings in the theory on D5-brane. They also show up as “freckles” in the gauge theory/sigma model correspondence⁶¹. We treat only the bosonic fields, but these could be a part of a supersymmetric multiplet.

A D3-brane can be surrounded by other branes as well. For example, in the Euclidean setup, a D-instanton could approach the D3-brane. In fact, unless the D-instanton is dissolved inside the brane, the combined system breaks supersymmetry⁸². The D3-D(-1) system can be rather simply described in terms of a noncommutative $U(1)$ gauge theory - the latter has instanton-like solutions⁷⁴.

More generally, one can have a stack of k Euclidean D3-branes with $ND(-1)$ -branes inside. This configuration will be described by charge N instantons in $U(k)$ gauge theory.

Let us work in four Euclidean space-time dimensions, $i, j = 1, 2, 3, 4$. As we said above, we shall look at the purely bosonic Yang-Mills theory on the space-time \mathcal{A}_θ with the coordinates functions x^i obeying the Heisenberg commutation relations:

$$[x^i, x^j] = \mathbf{i}\theta^{ij} \quad (57)$$

We assume that the metric on the space-time is Euclidean:

$$G_{ij} = \delta_{ij} \quad (58)$$

The action describing our gauge theory is given by:

$$S = -\frac{1}{4g_{\text{YM}}^2} \text{Tr} F \wedge *F + \frac{\mathbf{i}\vartheta}{4\pi} \text{Tr} F \wedge F \quad (59)$$

where g_{YM}^2 is the Yang-Mills coupling constant, ϑ is that gauge theory theta angle, and

$$F = \frac{1}{2} F_{ij} dx^i \wedge dx^j, \quad F_{ij} = [\nabla_i, \nabla_j] \quad (60)$$

Here the covariant derivatives act ∇_i in the free module F_k .

4.1 Instantons

The equations of motion following from (59) are:

$$\sum_i [\nabla_i, F_{ij}] = 0 \quad (61)$$

In general these equations are as hard to solve as the equations of motion of the ordinary non-abelian Yang-Mills theory. However, just like in the commutative case⁹, there are special solutions, which are simpler to analyze and which play a crucial role in the analysis of the quantum theory. These are the so-called (anti-)instantons. The (anti-)instantons solve the first order equation:

$$F_{ij} = \pm \frac{1}{2} \sum_{k,l} \varepsilon_{ijkl} F_{kl} \quad (62)$$

which imply (61) by virtue of the Bianci identity:

$$[\nabla_i, F_{jk}] + \text{cyclic permutations} = 0$$

These equations are easier to solve. The solutions are classified by the instanton charge:

$$N = -\frac{1}{8\pi^2} \text{Tr} F \wedge F \quad (63)$$

and the action (59) on such a solution is equal to:

$$S_N = 2\pi \mathbf{i} \tau N, \quad \tau = \frac{4\pi \mathbf{i}}{g_{\text{YM}}^2} + \frac{\vartheta}{2\pi} \quad (64)$$

Introduce the complex coordinates: $z_1 = x_1 + ix_2 = x_+$, $z_2 = x_3 + ix_4$. The instanton equations read:

$$F_{z_1 z_2} = 0, \quad F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = 0 \quad (65)$$

The first equation in (65) can be solved (locally) as follows:

$$A_{\bar{z}_a} = \xi^{-1} \bar{\partial}_{\bar{z}_a} \xi, \quad A_{z_a} = -\partial_{z_a} \xi \xi^{-1} . \quad (66)$$

with ξ a Hermitian matrix. Then the second equation in (65) becomes Yang's equation:

$$\sum_{a=1}^2 \bar{\partial}_{z_a} (\partial_{z_a} \xi^2 \xi^{-2}) = 0 . \quad (67)$$

In the noncommutative case this ansatz almost works globally (see below).

4.2 ADHM construction

In the commutative case all solutions to (62) with the finite action (59) are obtained via the so-called Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction⁵. If we are concerned with the instantons in the $U(k)$ gauge group, then the ADHM data consists of

1. the pair of the two complex vector spaces V and W of dimensions N and k respectively;
2. the operators: $B_1, B_2 \in \text{Hom}(V, V)$, and $I \in \text{Hom}(W, V)$, $J \in \text{Hom}(V, W)$;
3. the dual gauge group $G_N = U(N)$, which acts on the data above as follows:

$$B_\alpha \mapsto g^{-1} B_\alpha g, \quad I \mapsto g^{-1} I, \quad J \mapsto Jg \quad (68)$$

4. Hyperkähler quotient⁴⁷ with respect to the group (68). It means that one takes the set $X_{k,N} = \mu_r^{-1}(0) \cap \mu_c^{-1}(0)$ of the common zeroes of the three moment maps:

$$\begin{aligned} \mu_r &= [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J \\ \mu_c &= [B_1, B_2] + IJ \\ \bar{\mu}_c &= [B_2^\dagger, B_1^\dagger] + J^\dagger I^\dagger \end{aligned} \quad (69)$$

and quotients it by the action of G_N .

The claim of ADHM is that the points in the space $\mathcal{M}_{k,N} = X_{k,N}^\circ / G_N$ parameterize the solutions to (62) (for $\theta^{ij} = 0$) up to the gauge transformations. Here $X_{k,N}^\circ \subset X_{k,N}$ is the open dense subset of $X_{k,N}$ which consists of the solutions to $\vec{\mu} = 0$ such that their stabilizer in G_N is trivial. The explicit formula for the gauge field A_i is also known. Define the Dirac-like operator:

$$\mathcal{D}^+ = \begin{pmatrix} -B_2 + z_2 & B_1 - z_1 & I \\ B_1^\dagger - \bar{z}_1 & B_2^\dagger - \bar{z}_2 & J^\dagger \end{pmatrix} : V \otimes \mathbf{C}^2 \oplus W \rightarrow V \otimes \mathbf{C}^2 \quad (70)$$

Here z_1, z_2 denote the complex coordinates on the space-time:

$$z_1 = x_1 + ix_2, \quad z_2 = x_3 + ix_4, \quad \bar{z}_1 = x_1 - ix_2, \quad \bar{z}_2 = x_3 - ix_4$$

The kernel of the operator (70) is the x -dependent vector space $\mathcal{E}_x \subset V \otimes \mathbf{C}^2 \oplus W$. For generic x , \mathcal{E}_x is isomorphic to W . Let us denote by $\Psi = \Psi(x)$ this isomorphism. In plain words, Ψ is the fundamental solution to the equation:

$$\mathcal{D}^+ \Psi = 0, \quad \Psi : W \rightarrow V \otimes \mathbf{C}^2 \oplus W \quad (71)$$

If the rank of Ψ is x -independent (this property holds for generic points in \mathcal{M}), then one can normalize:

$$\Psi^\dagger \Psi = \text{Id}_k \quad (72)$$

which fixes Ψ uniquely up to an x -dependent $U(k)$ transformation $\Psi(x) \mapsto \Psi(x)g(x)$, $g(x) \in U(k)$. Given Ψ the anti-self-dual gauge field is constructed simply as follows:

$$\nabla_i = \partial_i + A_i, \quad A_i = \Psi^\dagger(x) \frac{\partial}{\partial x^i} \Psi(x) \quad (73)$$

The space of (B_0, B_1, I, J) for which $\Psi(x)$ has maximal rank for all x is an open dense subset $\mathcal{M}_{N,k} = X_{N,k}^\circ / G_N$ in \mathcal{M} . The rest of the points in $X_{N,k} / G_N$ describes the so-called *point-like* instantons. Namely, $\Psi(x)$ has maximal rank for all x but some finite set $\{x_1, \dots, x_l\}$, $l \leq k$. The (72) holds for $x \neq x_i$, $i = 1, \dots, l$, where the left hand side of (72) simply vanishes.

The noncommutative deformation of the gauge theory leads to the non-commutative deformation of the ADHM construction. It turns out to be very simple yet surprising. The same data V, W, B, I, J, \dots is used. The deformed ADHM equations are simply

$$\mu_r = \zeta_r, \mu_c = \zeta_c \quad (74)$$

where we have introduced the following notations. The Poisson tensor θ^{ij} entering the commutation relation $[x^i, x^j] = i\theta^{ij}$ can be decomposed into the self-dual and anti-self-dual parts θ^\pm . If we look at the commutation relations of the complex coordinates $z_1, z_2, \bar{z}_1, \bar{z}_2$ then the self-dual part of θ enters the following commutators:

$$[z_1, z_2] = -\zeta_c \quad [z_1, \bar{z}_1] + [z_2, \bar{z}_2] = -\zeta_r \quad (75)$$

It turns out that as long as $|\zeta| = \zeta_r^2 + \zeta_c \bar{\zeta}_c > 0$ one needs not to distinguish between $\tilde{X}_{N,k}$ and $\tilde{X}_{N,k}^\circ$, in other words the stabilizer of any point in $\tilde{X}_{N,k} = \mu_r^{-1}(-\zeta_r) \cap \mu_c^{-1}(-\zeta_c)$ is trivial. Then the resolved moduli space is $\tilde{\mathcal{M}}_{N,k} = \tilde{X}_{N,k} / G_N$.

By making an orthogonal rotation on the coordinates x^μ we can map the algebra \mathcal{A}_θ onto the sum of two copies of the Heisenberg algebra. These two algebras can have different values of “Planck constants”. Their sum is the norm of the self-dual part of θ , i.e. $|\zeta|$, and their difference is the norm of anti-self-dual part of θ :

$$[z_1, \bar{z}_1] = -\zeta_1, \quad [z_2, \bar{z}_2] = -\zeta_2 \quad (76)$$

where $\zeta_1 + \zeta_2 = |\theta^+|$, $\zeta_1 - \zeta_2 = |\theta^-|$. By the additional reflection of the coordinates, if necessary, one can make both ζ_1 and ζ_2 positive (however, one should be careful, since if the odd number of reflections was made, then the orientation of the space was changed and the notions of the instantons and anti-instantons are exchanged as well).

The next step in the ADHM construction was the definition of the isomorphism Ψ between the fixed vector space W and the fiber \mathcal{E}_x of the gauge bundle, defined as the kernel of the operator \mathcal{D}^+ . In the noncommutative setup one can also define the operator \mathcal{D}^+ by the same formula (70). It is a map between two free modules over \mathcal{A}_θ :

$$\mathcal{D}_x^+ : (V \otimes \mathbf{C}^2 \oplus W) \otimes \mathcal{A}_\theta \rightarrow (V \otimes \mathbf{C}^2) \otimes \mathcal{A}_\theta \quad (77)$$

which commutes with the right action of \mathcal{A}_θ on the free modules. Clearly,

$$\mathcal{E} = \text{Ker} \mathcal{D}^+$$

is a right module over \mathcal{A}_θ , for if $\mathcal{D}^+ s = 0$, then $\mathcal{D}^+(s \cdot a) = 0$, for any $a \in \mathcal{A}_\theta$.

\mathcal{E} is also a projective module, for the following reason. Consider the operator $\mathcal{D}^+ \mathcal{D}$. It is a map from the free module $V \otimes \mathbf{C}^2 \otimes \mathcal{A}_\theta$ to itself. Thanks to (74) this map actually equals to $\Delta \otimes \text{Id}_{\mathbf{C}^2}$ where Δ is the following map from the free module $V \otimes \mathcal{A}_\theta$ to itself:

$$\Delta = (B_1 - z_1)(B_1^\dagger - \bar{z}_1) + (B_2 - z_2)(B_2^\dagger - \bar{z}_2) + II^\dagger \quad (78)$$

We claim that Δ has no kernel, i.e. no solutions to the equation $\Delta v = 0$, $v \in V \otimes \mathcal{A}_\theta$. Recall the Fock space representation \mathcal{H} of the algebra \mathcal{A}_θ . The coordinates z_α, \bar{z}_α , obeying (76), with $\zeta_1, \zeta_2 > 0$, are represented as follows:

$$z_1 = \sqrt{\zeta_1} c_1^\dagger, \bar{z}_1 = \sqrt{\zeta_1} c_1, \quad z_2 = \sqrt{\zeta_2} c_2^\dagger, \bar{z}_2 = \sqrt{\zeta_2} c_2 \quad (79)$$

where $c_{1,2}$ are the annihilation operators and $c_{1,2}^\dagger$ are the creation operators for the two-oscillators Fock space

$$\mathcal{H} = \bigoplus_{n_1, n_2 \geq 0} \mathbf{C} |n_1, n_2\rangle$$

Let us assume the opposite, namely that there exists a vector $v \in V \otimes \mathcal{A}_\theta$ such that $\Delta v = 0$. Let us act by this vector on an arbitrary state $|n_1, n_2\rangle$ in \mathcal{H} . The result is the vector $\nu_{\bar{n}} \in V \otimes \mathcal{H}$ which must be annihilated by the operator Δ , acting in $V \otimes \mathcal{H}$ via (79). By taking the Hermitian inner product

of the equation $\Delta\nu_{\bar{n}} = 0$ with the conjugate vector $\nu_{\bar{n}}^\dagger$ we immediately derive the following three equations:

$$\begin{aligned}(B_2^\dagger - \bar{z}_2)\nu_{\bar{n}} &= 0 \\ (B_1^\dagger - \bar{z}_1)\nu_{\bar{n}} &= 0\end{aligned}\tag{80}$$

$$I^\dagger\nu_{\bar{n}} = 0$$

Using (74) we can also represent Δ_x in the form:

$$\Delta = (B_1^\dagger - \bar{z}_1)(B_1 - z_1) + (B_2^\dagger - \bar{z}_2)(B_2 - z_2) + J^\dagger J.\tag{81}$$

From this representation another triple of equations follows:

$$\begin{aligned}(B_2 - z_2)\nu_{\bar{n}} &= 0 \\ (B_1 - z_1)\nu_{\bar{n}} &= 0\end{aligned}\tag{82}$$

$$J\nu_{\bar{n}} = 0$$

Let us denote by $e_i, i = 1, \dots, N$ some orthonormal basis in V . We can expand $\nu_{\bar{n}}$ in this basis as follows:

$$\nu_{\bar{n}} = \sum_{i=1}^N e_i \otimes v_{\bar{n}}^i, \quad v_{\bar{n}}^i \in \mathcal{H}$$

The equations (80),(82) imply:

$$(B_\alpha)_j^i v_{\bar{n}}^j = z_\alpha v_{\bar{n}}^i, \quad (B_\alpha^\dagger)_j^i v_{\bar{n}}^j = \bar{z}_\alpha v_{\bar{n}}^i, \quad \alpha = 1, 2\tag{83}$$

in other words the matrices $B_\alpha, B_\alpha^\dagger$ form a finite-dimensional representation of the Heisenberg algebra which is impossible if either ζ_1 or $\zeta_2 \neq 0$. Hence $\nu_{\bar{n}} = 0$, for any $\bar{n} = (n_1, n_2)$ which implies that $v = 0$. Notice that this argument generalizes to the case where only one of $\zeta_\alpha \neq 0$.

Thus the Hermitian operator Δ is invertible. It allows to prove the following theorem: each vector ψ in the free module $(V \otimes \mathbf{C}^2 \oplus W) \otimes \mathcal{A}_\theta$ can be decomposed as a sum of two orthogonal vectors:

$$\psi = \Psi_\psi \oplus \mathcal{D}\chi_\psi, \quad \mathcal{D}^+\Psi_\psi = 0, \quad \chi_\psi \in (V \otimes \mathbf{C}^2) \otimes \mathcal{A}_\theta\tag{84}$$

where the orthogonality is understood in the sense of the following \mathcal{A}_θ -valued Hermitian product:

$$\langle \psi_1, \psi_2 \rangle = \text{Tr}_{V \otimes \mathbf{C}^2 \oplus W} \left(\psi_1^\dagger \psi_2 \right)$$

The component Ψ_ψ is annihilated by \mathcal{D}^+ , that is $\Psi_\psi \in \mathcal{E}$. The image of \mathcal{D} is another right module over \mathcal{A}_θ (being the image of the free module $(V \otimes \mathbf{C}^2) \otimes \mathcal{A}_\theta$):

$$\mathcal{E}' = \mathcal{D}(V \otimes \mathbf{C}^2 \otimes \mathcal{A}_\theta)$$

and their sum is a free module:

$$\mathcal{E} \oplus \mathcal{E}' = \mathcal{F} := (V \otimes \mathbf{C}^2 \oplus W) \otimes \mathcal{A}_\theta$$

hence \mathcal{E} is projective. It remains to give the expressions for Ψ_ψ, χ_ψ :

$$\chi_\psi = \frac{1}{\mathcal{D}^+ \mathcal{D}} \mathcal{D}^+ \psi, \quad \Psi_\psi = \Pi \psi, \quad \Pi = \left(1 - \mathcal{D} \frac{1}{\mathcal{D}^+ \mathcal{D}} \mathcal{D}^+ \right) \quad (85)$$

The noncommutative instanton is a connection in the module \mathcal{E} which is obtained simply by projecting the trivial connection on the free module \mathcal{F} down to \mathcal{E} . To get the covariant derivative of a section $s \in \mathcal{E}$ we view this section as a section of \mathcal{F} , differentiate it using the ordinary derivatives on \mathcal{A}_θ and project the result down to \mathcal{E} again:

$$\nabla s = \Pi \, ds \quad (86)$$

The curvature is defined through ∇^2 , as usual:

$$\nabla \nabla s = F \cdot s = d\Pi \wedge d\Pi \cdot s \quad (87)$$

where we used the following relations:

$$\Pi^2 = \Pi, \quad \Pi s = s \quad (88)$$

Let us now show explicitly that the curvature (87) is anti-self-dual, i.e.

$$[\nabla_i, \nabla_j] + \frac{1}{2} \epsilon_{ijkl} [\nabla_k, \nabla_l] = 0 \quad (89)$$

First we prove the following lemma: for any $s \in \mathcal{E}$:

$$d\Pi \wedge d\Pi s = \Pi d\mathcal{D} \frac{1}{\mathcal{D}^+ \mathcal{D}} d\mathcal{D}^+ s \quad (90)$$

Indeed,

$$d\Pi \wedge d\Pi = d \left(\mathcal{D} \frac{1}{\mathcal{D}^+ \mathcal{D}} \mathcal{D}^+ \right) \wedge d \left(\mathcal{D} \frac{1}{\mathcal{D}^+ \mathcal{D}} \mathcal{D}^+ \right),$$

$$\begin{aligned} d\left(\mathcal{D}\frac{1}{\mathcal{D}+\bar{\mathcal{D}}}\mathcal{D}^+\right) &= \Pi d\mathcal{D}\frac{1}{\mathcal{D}+\bar{\mathcal{D}}}\mathcal{D}^+ + \mathcal{D}\frac{1}{\mathcal{D}+\bar{\mathcal{D}}}d\mathcal{D}^+\Pi, \\ \mathcal{D}^+\Pi &= 0 \end{aligned}$$

hence

$$\begin{aligned} d\left(\mathcal{D}\frac{1}{\mathcal{D}+\bar{\mathcal{D}}}\mathcal{D}^+\right) \wedge d\left(\mathcal{D}\frac{1}{\mathcal{D}+\bar{\mathcal{D}}}\mathcal{D}^+\right) &= \\ &= \Pi d\mathcal{D}\frac{1}{\mathcal{D}+\bar{\mathcal{D}}}d\mathcal{D}^+\Pi + \mathcal{D}\frac{1}{\mathcal{D}+\bar{\mathcal{D}}}d\mathcal{D}^+\Pi d\mathcal{D}\frac{1}{\mathcal{D}+\bar{\mathcal{D}}}\mathcal{D}^+ \end{aligned}$$

and the second term vanishes when acting on $s \in \mathcal{E}$, while the first term gives exactly what the equation (90) states.

Now we can compute the curvature more or less explicitly:

$$F \cdot s = 2\Pi \begin{pmatrix} \frac{1}{\Delta}f_3 & \frac{1}{\Delta}f_+ & 0 \\ \frac{1}{\Delta}f_- & -\frac{1}{\Delta}f_3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot s \quad (91)$$

where f_3, f_+, f_- are the basic anti-self-dual two-forms on \mathbf{R}^4 :

$$f_3 = \frac{1}{2}(dz_1 \wedge d\bar{z}_1 - dz_2 \wedge d\bar{z}_2), \quad f_+ = dz_1 \wedge d\bar{z}_2, \quad f_- = d\bar{z}_1 \wedge dz_2 \quad (92)$$

Thus we have constructed the nonsingular anti-self-dual gauge fields over \mathcal{A}_θ . The interesting feature of the construction is that it produces the non-trivial modules over the algebra \mathcal{A}_θ , which are projective for any point in the moduli space. This feature is lacking in the $\zeta \rightarrow 0$, where it is spoiled by the point-like instantons. This feature is also lacking if the deformed ADHM equations are used for construction of gauge bundles directly over a commutative space. In this case it turns out that one can construct a torsion free sheaf over \mathbf{C}^2 , which sometimes can be identified with a holomorphic bundle. However, generically this sheaf will not be locally free. It can be made locally free by blowing up sufficiently many points on \mathbf{C}^2 , thereby effectively changing the topology of the space¹¹. The topology change is rather mysterious if we recall that it is purely gauge theory we are dealing with. However, in the supersymmetric case this gauge theory is an $\alpha' \rightarrow 0$ limit of the theory on a stack of Euclidean D3-branes. One could think that the topology changes reflect the changes of topology of D3-branes embedded into flat ambient space. This is indeed the case for monopole solutions, e.g.^{12,45,63,62}. It is not completely unimaginable possibility, but so far it has not been justified (besides from the fact that the DBI solutions^{82,91} are ill-defined without a blowup of the space). What makes this unlikely is the fact that the instanton backgrounds have no worldvolume scalars turned on.

At any rate, the noncommutative instantons constructed above are well-defined and nonsingular without any topology change.

Also note, that the formulae above define non-singular gauge fields for any $\theta \neq 0$. For $\theta^+ \neq 0$ these are instantons, for $\theta^- \neq 0$ they define anti-instantons (one needs to perform an orthogonal change of coordinates which reverses the orientation of \mathbf{R}^4).

The identifier Ψ

In the noncommutative case one can also try to construct the identifying map Ψ . It is to be thought as of the homomorphism of the modules over \mathcal{A} :

$$\Psi : W \otimes \mathcal{A}_\theta \rightarrow \mathcal{E}$$

The normalization (72), if obeyed, would imply the unitary isomorphism between the free module $W \otimes \mathcal{A}_\theta$ and \mathcal{E} . We can write: $\Pi = \Psi\Psi^\dagger$ and the elements s of the module \mathcal{E} can be cast in the form:

$$s = \Psi \cdot \sigma, \quad \sigma \in W \otimes \mathcal{A}_\theta \quad (93)$$

Then the covariant derivative can be written as:

$$\nabla s = \Pi d(\Psi \cdot \sigma) = \Psi\Psi^\dagger d(\Psi\sigma) = \Psi(d\sigma + A\sigma) \quad (94)$$

where

$$A = \Psi^\dagger d\Psi \quad (95)$$

just like in the commutative case. For $\text{Pf}(\theta) \neq 0$, after having introduced the “background independent” operators⁸¹ $D_i = i\theta_{ij}x^j + A_i$, we write:

$$D_i = i\Psi^\dagger \theta_{ij} x^j \Psi \quad (96)$$

5 Abelian instantons

Let us describe the case of $U(1)$ instantons in detail, i.e. the case $k = 1$ in our notations above. Let us assume that $\theta^+ \neq 0$. It is known, from⁶⁸, that for $\zeta_r > 0, \zeta_c = 0$ the solutions to the deformed ADHM equations have $J = 0$. Let us denote by V the complex Hermitian vector space of dimensionality N , where $B_\alpha, \alpha = 1, 2$ act. Then I is identified with a vector in V . We can choose our units and coordinates in such a way that $\zeta_r = 2, \zeta_c = 0$.

5.1 Torsion free sheaves on \mathbf{C}^2

Let us recall at this point the algebraic-geometric interpretation⁶⁸ of the space V and the triple (B_1, B_2, I) . The space $\tilde{X}_{N,1}$ parameterizes the rank one torsion free sheaves on \mathbf{C}^2 . In the case of \mathbf{C}^2 these are identified with the ideals \mathcal{I} in the algebra $\mathcal{O} \approx \mathbf{C}[z_1, z_2]$ of holomorphic functions on \mathbf{C}^2 , such that $V = \mathcal{O}/\mathcal{I}$ has dimension N . An ideal of the algebra \mathcal{O} is a subspace $\mathcal{I} \subset \mathcal{O}$, which is invariant under the multiplication by the elements of \mathcal{O} , i.e. if $g \in \mathcal{I}$ then $fg \in \mathcal{I}$ for any \mathcal{O} .

An example of such an ideal is given by the space of functions of the form:

$$f(z_1, z_2) = z_1^N g(z_1, z_2) + z_2 h(z_1, z_2)$$

The operators B_α are simply the operations of multiplication of a function, representing an element of V by the coordinate function z_α , and the vector I is the image in V of the constant function $f = 1$. In the example above, following¹¹ we identify V with $\mathbf{C}[z_1]/z_1^N$, the operator $B_2 = 0$, and in the basis $e_i = \sqrt{(i-1)!} z_1^{N-i}$ the operator B_1 is represented by a Jordan-type block: $B_1 e_i = \sqrt{2(i-1)} e_{i-1}$, and $I = \sqrt{2N} e_N$.

Conversely, given a triple (B_1, B_2, I) , such that the ADHM equations are obeyed the ideal \mathcal{I} is reconstructed as follows. The polynomial $f \in \mathbf{C}[z_1, z_2]$ belongs to the ideal, $f \in \mathcal{I}$ if and only if $f(B_1, B_2)I = 0$. Then, from the ADHM equations it follows that by acting on the vector I with polynomials in B_1, B_2 one generates the whole of V . Hence $\mathbf{C}[z_1, z_2]/\mathcal{I} \approx V$ and has dimension N , as required.

5.2 Identifier Ψ and projector P

Let us now solve the equations for the identifier: $\mathcal{D}^\dagger \Psi = 0$, $\Psi^\dagger \Psi = 1$. We decompose:

$$\Psi = \begin{pmatrix} \psi_+ \\ \psi_- \\ \xi \end{pmatrix} \quad (97)$$

where $\psi_\pm \in V \otimes \mathcal{A}_\theta$, $\xi \in \mathcal{A}_\theta$. The normalization (72) is now:

$$\psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- + \xi^\dagger \xi = 1 \quad (98)$$

It is convenient to work with rescaled matrices B : $B_\alpha = \sqrt{\zeta_\alpha} \beta_\alpha$, $\alpha = 1, 2$. The equation $\mathcal{D}^\dagger \Psi = 0$ is solved by the substitution:

$$\psi_+ = -\sqrt{\zeta_2}(\beta_2^\dagger - c_2)v, \quad \psi_- = \sqrt{\zeta_1}(\beta_1^\dagger - c_1)v \quad (99)$$

provided

$$\hat{\Delta}v + I\xi = 0, \quad \hat{\Delta} = \sum_{\alpha} \zeta_{\alpha}(\beta_{\alpha} - c_{\alpha}^{\dagger})(\beta_{\alpha}^{\dagger} - c_{\alpha}) \quad (100)$$

Fredholm's alternative states that the solution ξ of (100) must have the property, that for any $\nu \in \mathcal{H}, \chi \in V$, such that

$$\hat{\Delta}(\nu \otimes \chi) = 0, \quad (101)$$

the equation

$$(\nu^{\dagger} \otimes \chi^{\dagger}) I\xi = 0 \quad (102)$$

holds. It is easy to describe the space of all $\nu \otimes \chi$ obeying (101): it is spanned by the vectors:

$$e^{\sum \beta_{\alpha}^{\dagger} c_{\alpha}^{\dagger}} |0, 0\rangle \otimes e_i, \quad i = 1, \dots, N \quad (103)$$

where e_i is any basis in V . Let us introduce a Hermitian operator G in V :

$$G = \langle 0, 0 | e^{\sum \beta_{\alpha} c_{\alpha}} I I^{\dagger} e^{\sum \beta_{\alpha}^{\dagger} c_{\alpha}^{\dagger}} |0, 0\rangle \quad (104)$$

It is positive definite, which follows from the representation:

$$G = \langle 0, 0 | e^{\sum \beta_{\alpha} c_{\alpha}} \sum \zeta_{\alpha}(\beta_{\alpha}^{\dagger} - c_{\alpha})(\beta_{\alpha} - c_{\alpha}^{\dagger}) e^{\sum \beta_{\alpha}^{\dagger} c_{\alpha}^{\dagger}} |0, 0\rangle$$

and the fact that $\beta_{\alpha} - c_{\alpha}^{\dagger}$ has no kernel in $\mathcal{H} \otimes V$. Then define an element of the algebra \mathcal{A}_{θ}

$$P = I^{\dagger} e^{\sum \beta_{\alpha}^{\dagger} c_{\alpha}^{\dagger}} |0, 0\rangle G^{-1} \langle 0, 0 | e^{\sum \beta_{\alpha} c_{\alpha}} I \quad (105)$$

which obeys $P^2 = P$, i.e. it is a projector. Moreover, it is a projection onto an N -dimensional subspace in \mathcal{H} , isomorphic to V .

5.3 Dual gauge invariance

The normalization condition (72) is invariant under the action of the dual gauge group $G_N \approx U(N)$ on B_{α}, I . However, the projector P is invariant under the action of larger group - the complexification $G_N^{\mathbb{C}} \approx GL_N(\mathbb{C})$:

$$(B_{\alpha}, I) \mapsto (g^{-1} B_{\alpha} g, g^{-1} I), \quad (B_{\alpha}^{\dagger}, I^{\dagger}) \mapsto (g^{\dagger} B_{\alpha} g^{\dagger, -1}, I^{\dagger} g^{\dagger, -1}) \quad (106)$$

This makes the computations of P possible even when the solution to the $\mu_r = \zeta_r$ part of the ADHM equations is not known. The moduli space $\widetilde{\mathcal{M}}_{N,k}$ can be described both in terms of the hyperkahler reduction as above, or in terms of the quotient of the space of stable points $Y_{N,k}^s \subset \mu_c^{-1}(0)$ by the action of $G_N^{\mathbb{C}}$ (see ^{20,68} for related discussions). The stable points (B_1, B_2, I) are the ones where B_1 and B_2 commute, and generate all of V by acting on I : $\mathbb{C}[B_1, B_2]I = V$, i.e. precisely those triples which correspond to the codimension N ideals in $\mathbb{C}[z_1, z_2]$.

5.4 Instanton gauge field

Clearly, P annihilates ξ , thanks to (102). Let S be an operator in \mathcal{H} which obeys the following relations:

$$SS^\dagger = 1, \quad S^\dagger S = 1 - P \quad (107)$$

The existence of S is merely a reflection of the fact that as Hilbert spaces $\mathcal{H}_{\mathcal{I}} \approx \mathcal{H}$. So it just amounts to re-labeling the orthonormal bases in $\mathcal{H}_{\mathcal{I}}$ and \mathcal{H} to construct S . The operators S, S^\dagger were introduced in the noncommutative gauge theory context in⁴⁰ and played a prominent role in the constructions of various solutions of noncommutative gauge theories, e.g. in^{40,1,31,42,41,32}.

Now, $\hat{\Delta}$ restricted at the subspace $S^\dagger \mathcal{H} \otimes I \subset \mathcal{H} \otimes V$, is invertible. We can now solve (100) as follows:

$$\xi = \Lambda^{-\frac{1}{2}} S^\dagger, v = -\frac{1}{\hat{\Delta}} I \xi \quad (108)$$

where

$$\Lambda = 1 + I^\dagger \frac{1}{\hat{\Delta}} I \quad (109)$$

Λ is not an element of \mathcal{A}_θ , but Λ^{-1} and ΛS^\dagger are. Finally, the gauge fields can be written as:

$$D_\alpha = \sqrt{\frac{1}{\zeta_\alpha}} S \Lambda^{-\frac{1}{2}} c_\alpha \Lambda^{\frac{1}{2}} S^\dagger, \quad \bar{D}_{\bar{\alpha}} = -\sqrt{\frac{1}{\zeta_\alpha}} S \Lambda^{\frac{1}{2}} c_\alpha^\dagger \Lambda^{-\frac{1}{2}} S^\dagger \quad (110)$$

If $\zeta_1 \zeta_2 = 0$ then the formula (110) must be modified in an interesting way. We leave this as an exercise. Notice that if $S^\dagger S$ was equal to 1 then the expressions (110) had the Yang form (66).

5.5 Ideal meaning of P

We can explain the meaning of P in an invariant fashion, following²⁹. Consider the ideal \mathcal{I} in $\mathbf{C}[z_1, z_2]$, corresponding to the triple (B_1, B_2, I) as explained above. Any polynomial $f \in \mathcal{I}$ defines a vector $f(\sqrt{\zeta_1} c_1^\dagger, \sqrt{\zeta_2} c_2^\dagger) |0, 0\rangle$ and their totality span a subspace $\mathcal{H}_{\mathcal{I}} \subset \mathcal{H}$ of codimension N . The operator P is simply an orthogonal projection onto the complement to $\mathcal{H}_{\mathcal{I}}$. The fact \mathcal{I} is an ideal in $\mathbf{C}[z_1, z_2]$ implies that $c_\alpha^\dagger(\mathcal{H}_{\mathcal{I}}) \subset \mathcal{H}_{\mathcal{I}}$, hence:

$$c_\alpha^\dagger S^\dagger \eta = S^\dagger \eta'$$

for any $\eta \in \mathcal{A}_\theta$, and also $\Lambda^{-\frac{1}{2}} S^\dagger = S^\dagger \eta''$ for some $\eta', \eta'' \in \mathcal{A}_\theta$.

Notice that the expressions (110) are well-defined. For example, the $\bar{D}_{\bar{\alpha}}$ component contains a dangerous piece $\Lambda^{\frac{1}{2}} c_{\alpha}^{\dagger} \dots$ in it. However, in view of the previous remarks it is multiplied by S^{\dagger} from the right and therefore well-defined indeed.

5.6 Charge one instanton

In this case: $I = \sqrt{2}$, one can take $B_{\alpha} = 0$, $\hat{\Delta} = \sum \zeta_{\alpha} n_{\alpha}$, and in addition we shall assume that $\zeta_1 \zeta_2 \neq 0$.

$$\Lambda = \frac{M+2}{M}$$

$M = \sum_{\alpha} \zeta_{\alpha} n_{\alpha}$, $\sum_{\alpha} \zeta_{\alpha} = 2$. Let us introduce the notation $N = n_1 + n_2$. For the pair $\bar{n} = (n_1, n_2)$ let $\rho_{\bar{n}} = \frac{1}{2}N(N-1) + n_1$. The map $\bar{n} \leftrightarrow \rho_{\bar{n}}$ is one-to-one. Let $S^{\dagger}|\rho_{\bar{n}}\rangle = |\rho_{\bar{n}} + 1\rangle$. Clearly, $SS^{\dagger} = 1$, $S^{\dagger}S = 1 - |0, 0\rangle\langle 0, 0|$.

The formulae (110) are explicitly non-singular. Let us demonstrate the anti-self-duality of the gauge field (110) in this case.

$$\begin{aligned} \sum_{\alpha} D_{\alpha} \bar{D}_{\bar{\alpha}} &= -S \frac{1}{\zeta_{\alpha}} (n_{\alpha} + 1) \frac{M}{M+2} \frac{M+2+\zeta_{\alpha}}{M+\zeta_{\alpha}} S^{\dagger} \\ \sum_{\alpha} \bar{D}_{\bar{\alpha}} D_{\alpha} &= S \frac{1}{\zeta_{\alpha}} n_{\alpha} \frac{M-\zeta_{\alpha}}{M+2-\zeta_{\alpha}} \frac{M+2}{M} S^{\dagger} \end{aligned}$$

A simple calculation shows:

$$\sum_{\alpha} [D_{\alpha}, \bar{D}_{\bar{\alpha}}] = -\frac{2}{\zeta_1 \zeta_2} = -\left(\frac{1}{\zeta_1} + \frac{1}{\zeta_2}\right), \quad [D_{\alpha}, D_{\beta}] = 0, \quad (111)$$

hence

$$\sum_{\alpha} F_{\alpha\bar{\alpha}} = 0 \quad (112)$$

as

$$i \sum_{\alpha} \theta_{\alpha\bar{\alpha}} = \frac{1}{\zeta_1} + \frac{1}{\zeta_2}$$

This is a generalization of the charge one instanton constructed in ⁷⁴, written in the explicitly non-singular gauge. The explicit expressions for higher charge instantons are harder to write, since the solution of the deformed ADHM equations is not known in full generality. However, in the case of the so-called “elongated” instantons¹¹ the solution can be written down rather explicitly, with the help of Charlier polynomials. In the $U(2)$ case it is worth trying to apply the results of⁵⁵ for the studies of the instantons of charges ≤ 3 .

5.7 Remark on gauges

The gauge which was chosen in the examples considered in⁷⁴ and subsequently adopted in^{29,38} had $\xi = \xi^\dagger$. It was shown in²⁹ that this gauge does not actually lead to the canonically normalized identifier Ψ : one had $\Psi^\dagger \Psi = 1 - P$. This gauge is in some sense an analogue of 't Hooft singular gauge for commutative instantons: it leads to singular formulae, if the gauge field is considered to be well-defined globally over \mathcal{A}_θ (similar observations were made previously in³⁰). However, as we showed above, there are gauges in which the gauge field is globally well-defined, non-singular, and anti-self-dual. They simply have $\xi \neq \xi^\dagger$.

6 Monopoles in noncommutative gauge theories

6.1 Realizations of monopoles via D-branes

Another interesting BPS configuration of D-branes is that of a D-string that ends on a D3-brane. The endpoint of the D-string is a magnetic charge for the gauge field on the D3-brane. In the commutative case, in the absence of the B -field, the D-string is a straight line, orthogonal to the D3-brane. It projects onto the D3-brane in the form of a singular source, located at the point where the D-string touches the D3-brane. From the point of view of the D3-brane this is a Dirac monopole, with energy density that diverges at the origin.

The situation changes drastically when the B -field is turned on. One can trade a constant background B -field with spatial components for a constant background magnetic field. The latter pulls the magnetic monopoles with the constant force. As a consequence, the D-string bends⁴⁵, in such a way that its tension compensates the magnetic force. It projects to the D3-brane as a half-line with finite tension. It is quite fascinating to see that the shadow of this string is seen by the noncommutative gauge theory. The $U(1)$ noncommutative gauge theory with adjoint Higgs field has a monopole solution³⁹, which is everywhere non-singular, and whose energy density is peaked along a half-line, making up a semi-infinite string. The non-singularity of the solution is non-perturbative in θ and cannot be seen by the expansion in θ around the Dirac monopole⁵¹.

This analytic solution extends to the case of $U(2)$ noncommutative gauge theory. In this case one finds strings of finite extent, according to the brane picture⁴⁵, which has the D-string suspended between two D3-branes separated by a finite distance. The solitons in $U(1)$ theory were localized in the noncommutative directions, but generically occupied all of the commutative space, corresponding to (semi)infinite $D(p-2)$ -branes, immersed in a Dp -brane, or piercing it. In the case of several Dp -branes we shall describe solitons which, although they have finite extent in the commutative directions, are nevertheless localized and look like codimension three objects when viewed from far away. The simplest such object is the monopole in the noncommutative $U(2)$ gauge theory, i.e. the theory on a stack of two separated D3-branes in the Seiberg-Witten limit⁸².

The fact that all the fields involved are non-singular, and that the solution is in fact a solution to the noncommutative version of the Bogomolny equations everywhere, shows that the string in the monopole solution is an intrinsic object of the gauge theory. As such, one could expect that the noncommutative gauge theory describes strings as well. In fact, the noncommutative gauge theory has solutions, describing infinite magnetic flux strings, whose fluctuations match

with those of D-strings, located anywhere in the ten dimensional space around the D3-branes⁴¹.

6.2 Monopole equations

If we look for the solutions to (62), that are invariant under translations in the 4'th direction then we will find the monopoles of the gauge theory with an adjoint scalar Higgs field, where the role of the Higgs field is played by the component A_4 of the gauge field. The equations (62) in this case are called the Bogomolny equations, and they can be analyzed in the commutative case via Nahm's ansatz⁶⁴.

For the x_4 -independent field configurations the action (24) produces the energy functional for the coupled gauge-adjoint Higgs system:

$$\mathcal{E} = \frac{2\pi\theta}{4g_{\text{YM}}^2} \int dx_3 \sqrt{\det G} \text{Tr} \left(-G^{ii'} G^{jj'} F_{ij} \star F_{i'j'} + 2 G^{ij} \nabla_i \Phi \star \nabla_j \Phi \right) \quad (113)$$

where for the sake of generality we have again introduced a constant metric G_{ij} . As before, the factor $2\pi\theta$ comes from relating the integral over x^1, x^2 to the trace over the Fock space which replaces the integration over the non-commutative part of the three dimensional space. The trace also includes the summation over the color indices, if they are present (for several D3-branes). In terms of the three dimensional gauge fields and the adjoint Higgs the Bogomolny equations have the form:

$$\nabla_i \Phi = \pm B_i, \quad i = 1, 2, 3. \quad (114)$$

where (49) in the case of generic metric G has the form:

$$B_i = \frac{\mathbf{i}}{2} \varepsilon_{ijk} G^{jj'} G^{kk'} \sqrt{G} F_{j'k'}$$

As in the ordinary, commutative case, one can rewrite (113) as a sum of a total square and a total derivative:

$$\begin{aligned} \mathcal{E} = & \frac{\pi\theta}{g_{\text{YM}}^2} \int dx_3 \sqrt{G} G^{ij} \text{Tr} (\nabla_i \Phi \pm B_i) \star (\nabla_j \Phi \pm B_j) \\ & \mp \partial_j \left[\sqrt{G} G^{ij} \text{Tr} (B_i \star \Phi + \Phi \star B_i) \right] \end{aligned} \quad (115)$$

The total derivative term depends only on the boundary conditions. So, to minimize the energy given boundary conditions we should solve the *Bogomolny* equations (114).

6.3 Nahm's construction

Ordinary monopoles

To begin with, we review the techniques which facilitate the solution of the ordinary Bogomolny equations:

$$\nabla_i \Phi + B_i = 0, \quad i = 1, 2, 3 \quad (116)$$

They are supplemented with the boundary condition that at the spatial infinity the norm of the Higgs field approaches a constant, corresponding to the Higgs vacuum. In the case of $SU(2)$ this means that locally on the two-sphere at infinity:

$$\phi(x) \sim \text{diag} \left(\frac{a}{2}, -\frac{a}{2} \right) . \quad (117)$$

The solutions are classified by the magnetic charge N . By virtue of the equation (116) the monopole charge can be expressed as the winding number which counts how many times the two-sphere \mathbf{S}_∞^2 at infinity is mapped to the coset space $SU(2)/U(1) \approx \mathbf{S}^2$ of the abelian subgroups of the gauge group. We shall present a general discussion of the charge N monopoles in the gauge group $U(k)$, where k will be either 1 or 2.

The approach to the solution of (116) is via the modification of the ADHM construction. After all, (116) are also instanton equations, with different asymptotic conditions on the gauge fields. The appropriate modification of the ADHM construction was found by Nahm. Nahm⁶⁴ constructs solutions to the monopole equations as follows. Consider the matrix differential operator on the interval I with the coordinate z :

$$-\mathbf{i}\Delta = \partial_z + \mathcal{T}_i \sigma_i, \quad (118)$$

where

$$\mathcal{T}_i = T_i(z) + x_i . \quad (119)$$

x_i are the coordinates in the physical space \mathbf{R}^3 , and the $N \times N$ matrices $T_i(z) = T_i^\dagger(z)$ obey Nahm's equations:

$$\partial_z T_i = \frac{\mathbf{i}}{2} \varepsilon_{ijk} [T_j, T_k] , \quad (120)$$

with certain boundary conditions.

The range of the coordinate z depends on the specifics of the problem we are to address. For the $SU(2)$ monopoles with the asymptotics(117) we take $I = (-a/2, a/2)$ where a is given in (117). For the $U(1)$ Dirac monopoles

we would take $I = (-\infty, 0)$. At z approaches the boundary of I , $z \rightarrow z_0$ we require that :

$$T_i \sim \frac{t_i}{z - z_0} + \text{reg.}, \quad [t_i, t_j] = \mathbf{i}\varepsilon_{ijk}t_k, \quad (121)$$

i.e. the residues t_i must form a N -dimensional representation of $SU(2)$ (irreducible if the solution is to be non-singular).

Then one looks for the fundamental solution to the equation:

$$-i\Delta^\dagger\Psi(z) = \partial_z\Psi - \mathcal{T}_i\sigma_i\Psi = 0, \quad (122)$$

where

$$\Psi = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix},$$

and Ψ_\pm are $N \times k$ matrices, which must be finite at ∂I and normalized so that:

$$\int_I dz \Psi^\dagger \Psi = \mathbf{1}_{k \times k}. \quad (123)$$

Then:

$$A_i = \int_I dz \Psi^\dagger \partial_i \Psi, \quad (124)$$

$$\Phi = \int_I dz z \Psi^\dagger \Psi.$$

Notice that for $k = 2$ the interval I could be (a_1, a_2) instead of $(-a/2, +a/2)$. The only formula that is not invariant under shifts of z is the expression (124) for Φ . By shifting Φ by a scalar $(a_1 + a_2)/2$ we can make it traceless and map I back to the form we used above.

Abelian ordinary monopoles

In the case $k = 1$ Nahm's equations describe Dirac monopoles. Take $I = (-\infty, 0)$. The equation (116) becomes simply the condition that the abelian monopole has a magnetic potential Φ , which must be harmonic. Let us find this harmonic function. The matrices T_i can be taken to have the following form:

$$T_i(z) = \frac{t_i}{z}, \quad [t_i, t_j] = \mathbf{i}\varepsilon_{ijk}t_k, \quad (125)$$

where t_i form an irreducible spin j representation of $SU(2)$. Let $V \approx \mathbf{C}^N$, $N = 2j + 1$, be the space of this representation. The matrices Ψ_\pm are now V -valued.

By an $SU(2)$ rotation we can bring the three-vector x_i to the form $(0, 0, r)$, i.e. $x_1 = x_2 = 0, x_3 > 0$. Then in this basis:

$$\Psi_- = 0, \quad \Psi_+ = \frac{\sqrt{2r}}{\sqrt{(N-1)!}} (2rz)^j e^{rz} |j\rangle, \quad (126)$$

where $|j\rangle \in V$ is the highest spin state in V . From this we get the familiar formula for the singular Higgs field

$$\Phi = -\frac{N}{2r}. \quad (127)$$

corresponding to N Dirac monopoles sitting on top of each other.

Nonabelian ordinary monopoles.

We now take $k = 2, N = 1$. Let $H = \mathbf{C}^k$ be the Chan-Paton space, the fundamental representation of the gauge group. Let e_0, e_1 be the orthonormal basis in H . Again, for $N = 1$ the analysis of the equation (120) is simple: $T_i = 0$. We can take $a_{\pm} = \pm \frac{a}{2}$ and

$$\Psi = \begin{pmatrix} (\partial_z + x_3) v \\ (x_1 + \mathbf{i}x_2) v \end{pmatrix}, \quad \partial_z^2 v = r^2 v, \quad r^2 = \sum_i x_i^2. \quad (128)$$

The condition that Ψ is finite at both ends of the interval allows for two solutions of (70) in the form of (128):

$$v = e^{\pm rz},$$

which after imposing the normalization condition, (186), leads to:

$$\Psi = \frac{1}{\sqrt{2\sinh(ra)}} \left[e^{rz} \begin{pmatrix} \sqrt{r+x_3} \\ \frac{x_+}{\sqrt{r+x_3}} \end{pmatrix} \otimes e_0 + e^{-rz} \begin{pmatrix} -\sqrt{r-x_3} \\ \frac{x_-}{\sqrt{r-x_3}} \end{pmatrix} \otimes e_1 \right],$$

where $x_{\pm} = x_1 \pm \mathbf{i}x_2$.

In particular,

$$\Phi = \frac{1}{2} \left(\frac{a}{\tanh(ra)} - \frac{1}{r} \right) \sigma_3.$$

This is famous 't Hooft-Polyakov monopole of the higgsed $SU(2)$ gauge theory.

Nahm's equations from the D-string point of view

The meaning of the Nahm's equations becomes clearer in the D-brane realization of gauge theory and the D-string construction of monopoles. The endpoint of a fundamental string touching a D3-brane looks like an electric charge for the $U(1)$ gauge field on the brane. By S-duality, a D-string touching a D3-brane creates a magnetic monopole. If one starts with two parallel D3-branes, separated by distance a between them, one is studying the $U(2)$ gauge theory, Higgsed down to $U(1) \times U(1)$, where the vev of the Higgs field is

$$\Phi = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$

One can suspend a D-string between these two D3-branes, or a collection of k parallel D-strings. These would correspond to a charge k magnetic monopole in the Higgsed $U(2)$ theory. The BPS configurations of these D-strings are described the corresponding self-duality equations in the 1+1 dimensional $U(k)$ gauge theory on the worldsheet of these strings¹⁹, z being the spatial coordinate along the D-string. The equations (120) are exactly these BPS equations. The presence of the D3-branes is reflected in the boundary conditions (121). The matrices T_i correspond to the “matrix” transverse coordinates X^i , $i = 1, 2, 3$ to the D-strings, which lie within D3-branes. One can also consider the BPS configurations of semi-infinite D-strings, in which case the parameter z lives on a half-line. For example a collection of N D-strings ending on the D3-brane forms the so-called BIon¹⁶, described by the solution(125).

Old point of view

The old-fashioned point of view at the equations (122),(71) is that they are the equations obeyed by the kernel of the family of the Dirac operators in the background of the gauge/Higgs fields obeying self-duality condition^{64,17}. This interpretation also holds in the noncommutative case^{74,73}.

Noncommutative monopole equations

Now let us study the solutions to the Bogomolny equations for a gauge theory on a noncommutative three dimensional space. As before, we assume the Poisson structure (θ) which deforms the multiplication of the functions to be constant. Then there is essentially a unique choice of coordinate functions x_1, x_2, x_3 such that the commutation relations between them are as follows (cf. (3)):

$$[x_1, x_2] = -i\theta, \theta > 0 \quad [x_1, x_3] = [x_2, x_3] = 0 .$$

This algebra defines noncommutative \mathbf{R}^3 , which we still denote by \mathcal{A}_θ . Introduce the creation and annihilation operators c, c^\dagger :

$$c = \frac{1}{\sqrt{2\theta}}(x_1 - \mathbf{i}x_2), \quad c^\dagger = \frac{1}{\sqrt{2\theta}}(x_1 + \mathbf{i}x_2), \quad (129)$$

that obey

$$[c, c^\dagger] = 1.$$

We wish to solve Bogomolny equations(116), which can also be written as:

$$[D_i, \Phi] = \frac{\mathbf{i}}{2}\varepsilon_{ijk}[D_j, D_k] - \delta_{i3}\frac{1}{\theta}, \quad (130)$$

where Φ and D_i , $i = 1, 2, 3$ are the x_3 -dependent operators in $\mathcal{H} \otimes H$. Now the relation between D_i and A_i is as follows:

$$D_3 = \partial_3 + A_3, \quad D_\alpha = \mathbf{i}\theta^{-1}\varepsilon_{\alpha\beta}x^\beta + A_\alpha, \quad \alpha, \beta = 1, 2 \quad (131)$$

Noncommutative Nahm equations

We proceed *a la* ⁷⁴ by repeating the procedure that worked in the ADHM instanton case, namely we relax the condition that x_i 's commute but insist on the equation (120) with T_i replaced by the relevant matrices $\mathcal{T}_i = T_i + x_i$. Then the equation (120) on T_i is modified:

$$\partial_z T_i = \frac{\mathbf{i}}{2}\varepsilon_{ijk}[T_j, T_k] + \delta_{i3}\theta. \quad (132)$$

It is obvious that, given a solution $T_i(z)$ of the original Nahm equations(120), it is easy to produce a solution of the noncommutative ones:

$$T_i(z)^{\text{nc}} = T_i(z) + \theta z \delta_{i3}. \quad (133)$$

From this it follows that, unlike the case of instanton moduli space, the monopole moduli space *does not change under noncommutative deformation*.

Notice the similarity of (132) and (130), which becomes even more striking if we take into account that the spectral meaning of the coordinate z as the eigenvalue of the operator Φ . This similarity is explained in the framework of *noncommutative reciprocity*⁷³, generalizing the commutative reciprocity^{17,71}

The deformation(132) is exactly what one gets by looking at the D-strings suspended between the D3-branes^{6,63} (or a semi-infinite D-string with one end on a D3-brane) in the presence of a B -field. One gets the deformation:

$$[X^i, X^j] \rightarrow [X^i, X^j] - \mathbf{i}\theta^{ij} = [T_i, T_j] - \frac{1}{2}\theta\epsilon_{ij3} \quad (134)$$

The reason why θ^{ij} , instead of B_{ij} , appears on the right hand side of (134) is rather simple. By applying T-duality in the directions x_1, x_2, x_3 we could map the D-string into the D4-brane. The matrices X^1, X^2, X^3 become the components A_1, A_2, A_3 of the gauge field on the D4-brane worldvolume, and the B -field would couple to these gauge fields via the standard coupling $F_{ij} - \hat{B}_{ij}$, where \hat{B}_{ij} is the T-dualized B -field. It remains to observe that $\hat{B}_{ij} = \theta^{ij}$, since the T-dualized indices \hat{i} label the coordinates on the space, dual to that of x_i 's.

6.4 Solving Nahm's equations for noncommutative monopoles

To solve (132) we imitate the approach for the charge $N = 1$ ordinary monopole by taking

$$T_{1,2} = 0, \quad T_3 = \theta z. \quad (135)$$

To solve (70) for Ψ we introduce the operators b, b^\dagger :

$$b = \frac{1}{\sqrt{2\theta}}(\partial_z + x_3 + \theta z), \quad b^\dagger = \frac{1}{\sqrt{2\theta}}(-\partial_z + x_3 + \theta z), \quad (136)$$

which obey the oscillator commutation relations:

$$[b, b^\dagger] = [c, c^\dagger] = 1. \quad (137)$$

Introduce the *superpotential*

$$W = x_3 z + \frac{1}{2}\theta z^2, \quad (138)$$

so that $b = \frac{1}{\sqrt{2\theta}}e^{-W}\partial_z e^W$, $b^\dagger = -\frac{1}{\sqrt{2\theta}}e^W\partial_z e^{-W}$. Then equation (70) becomes:

$$\begin{aligned} b^\dagger \Psi_+ + c \Psi_- &= 0 \\ c^\dagger \Psi_+ - b \Psi_- &= 0. \end{aligned} \quad (139)$$

It is convenient to solve first the equation

$$\begin{aligned} b^\dagger \epsilon_+ + c \epsilon_- &= 0, \\ c^\dagger \epsilon_+ - b \epsilon_- &= 0, \end{aligned} \quad (140)$$

with $\epsilon_{\pm}(z, x_3) \in \mathcal{H}$. The number of solutions to (140) depends on what the interval I is. If $I = (a_-, a_+)$ is finite then (140) has the following solutions:

$$\begin{aligned}\epsilon^\alpha &= \begin{pmatrix} \epsilon_+^\alpha \\ \epsilon_-^\alpha \end{pmatrix}, \quad \alpha = 0, 1 \\ \epsilon_0^0 &= \begin{pmatrix} 0 \\ \frac{1}{\sqrt{\zeta_0}} e^{-W} |0\rangle \end{pmatrix}, \quad \zeta_0 = \int_{a_-}^{a_+} dz e^{-2W}, \quad \epsilon_0^1 = 0 \\ \epsilon_n^\alpha &= \begin{pmatrix} b \beta_n^\alpha |n-1\rangle \\ \sqrt{n} \beta_n^\alpha |n\rangle \end{pmatrix}, \quad n > 0,\end{aligned}\tag{141}$$

where e_0, e_1 will be the basis vectors in the two dimensional Chan-Paton space. The functions β_n^0, β_n^1 solve

$$(b^\dagger b + n) \beta_n^\alpha = 0,\tag{142}$$

and are required to obey the following boundary conditions:

$$\begin{aligned}b \beta_n^1(a_+) &= 0, & \beta_n^0(a_-) &= 0 \\ \beta_n^1 b \beta_n^1(a_-) &= -1, & \beta_n^0 b \beta_n^0(a_+) &= 1\end{aligned}\tag{143}$$

which together with (142) imply that:

$$\int_{a_-}^{a_+} dz (\epsilon_n^\alpha)^\dagger \epsilon_m^\gamma = \delta^{\alpha\gamma} \delta_{mn},\tag{144}$$

A solution to (139) is given by:

$$\Psi = \sum_{n \geq 0, \alpha=0,1} \epsilon_n^\alpha \cdot \langle n - \alpha | \otimes e_\alpha^\dagger.\tag{145}$$

and by virtue of (144) it obeys (186). All other solutions to (139), which are normalizable on $I = (a_-, a_+)$ are gauge equivalent to (145).

If $I = (-\infty, 0)$ then the number of solutions to (140) is roughly halved.

$$\begin{aligned}\epsilon &= \begin{pmatrix} \epsilon_+ \\ \epsilon_- \end{pmatrix}, \\ \epsilon_0 &= \begin{pmatrix} 0 \\ \frac{1}{\sqrt{\zeta_0}} e^{-W} |0\rangle \end{pmatrix}, \quad \zeta_0 = \int_{-\infty}^0 dz e^{-2W}, \\ \epsilon_n &= \begin{pmatrix} b \beta_n |n-1\rangle \\ \sqrt{n} \beta_n |n\rangle \end{pmatrix}, \quad n > 0,\end{aligned}\tag{146}$$

The functions β_n solve

$$(b^\dagger b + n) \beta_n = 0,\tag{147}$$

and are required to obey the following boundary conditions:

$$\beta_n b \beta_n(0) = 1, \quad \beta_n(z) \rightarrow 0, \quad z \rightarrow -\infty \quad (148)$$

which together with (147) imply that:

$$\int_{-\infty}^0 dz (\varepsilon_n)^\dagger \varepsilon_m = \delta_{mn}, \quad (149)$$

A solution to (139) is given by:

$$\Psi = \sum_{n \geq 0} \varepsilon_n \cdot |n\rangle. \quad (150)$$

and by virtue of (149) it obeys (186). All other solutions to (139) which are normalizable on $I = (-\infty, 0)$ are gauge equivalent to (145).

Generating solutions of the auxiliary problem: $U(1)$ case

To get the solutions to (147) solve first the equation for $n = 1$ and then act on it by b^{n-1} to generate the solution for higher n 's. The result is:

$$\beta_n(z) = \frac{\zeta_{n-1}(x_3 + \frac{1}{2}z)}{\sqrt{\zeta_n(x_3)\zeta_{n-1}(x_3)}} \quad (151)$$

where (we set $2\theta = 1$):

$$\zeta_n(z) = \int_0^\infty p^n e^{2pz - \frac{p^2}{2}} dp \quad (152)$$

The functions ζ_n obey³⁹:

$$\begin{aligned} \zeta_{n+1}(z) &= 2z\zeta_n(z) + n\zeta_{n-1}(z), \\ \partial_z \zeta_n &= 2\zeta_{n+1}, \\ \zeta_n(0) &= 2^{\frac{n-1}{2}} \left(\frac{n-1}{2}\right)! \end{aligned} \quad (153)$$

We have explicitly constructed β_n and thus Ψ_\pm , from which we can determine, using (124), the Higgs and gauge fields. To this end:

•Show that:

$$\int_{-\infty}^0 \beta_n b \beta_n = \zeta_{n-1} \zeta_{n+1} - \zeta_n^2 = n \zeta_{n-1}^2 - (n-1) \zeta_n \zeta_{n-2}, \quad n > 0 \quad (154)$$

Introduce the functions $\xi, \tilde{\xi}$ and $\eta = \tilde{\xi}^2$:

$$\tilde{\xi}(n) = \sqrt{\frac{\zeta_n}{\zeta_{n+1}}}, \quad \eta(n) = \frac{\zeta_n}{\zeta_{n+1}}, \quad \xi(n) = \sqrt{\frac{n\zeta_{n-1}}{\zeta_n}}. \quad (155)$$

We will need the asymptotics of these functions for large x_3 . Let $r_n^2 = x_3^2 + n$. For $r_n + x_3 \rightarrow \infty$ we can estimate the integral in (152) by the saddle point method. The saddle point and the approximate values of ζ_n and η_n are:

$$\begin{aligned} \bar{p} &= x_3 + r_n \\ \zeta_n &\sim \sqrt{\frac{\pi}{r_n}} (x_3 + r_n)^{n+\frac{1}{2}} e^{\frac{1}{2}(x_3+r_n)(3x_3-r_n)} \\ \eta_n &\sim \frac{1}{x_3 + r_{n+1}} \left(1 + \frac{1}{4r_n^2} + \dots \right). \end{aligned} \quad (156)$$

We shall also need:

$$\zeta_0(z) \sim \begin{cases} \sqrt{2\pi} e^{2z^2}, & z \rightarrow +\infty \\ \frac{1}{2}|z|^{-1}, & z \rightarrow -\infty \end{cases} \quad (157)$$

Generating solutions of the auxiliary problem: $U(2)$ case

It is easy to generate the solutions to (142): first of all,

$$f_n(z) = b^{n-1}(e^W) = e^{W(z)} h_{n-1}(2x_3 + z), \quad h_k(u) = e^{-\frac{u^2}{4}} \frac{d^k}{du^k} e^{\frac{u^2}{4}}$$

is a solution . Then

$$\hat{f}_n = f_n(z) \int_z^z \frac{du}{f_n(u)^2}$$

is the second solution. Notice that, for k even, $h_k(u) > 0$ for all u and, for k odd, the only zero of $h_k(u)$ is at $u = 0$, and $h_k(u)/u > 0$ for all u . Therefore, $\hat{h}_k(z)$ is well-defined for all z .

Consequently,

$$\begin{aligned} \beta_n^0(z) &= \tilde{\nu}_n f_n(z) \int_{a_-}^z \frac{du}{f_n(u)^2}, \\ \beta_n^1 &= \nu_n \left(-\frac{1}{nf_{n+1}(z)} + f_n(z) \int_z^{a_+} \frac{du}{f_{n+1}(u)^2} \right), \end{aligned} \quad (158)$$

where

$$\begin{aligned}\nu_n^{-2} &= \left(f_n(a_-)f_{n+1}(a_-) \int_{a_-}^{a_+} \frac{du}{f_{n+1}^2(u)} - \frac{1}{n} \right) \int_{a_-}^{a_+} \frac{du}{f_{n+1}^2(u)} , \\ \tilde{\nu}_n^{-2} &= \left(f_n(a_+)f_{n+1}(a_+) \int_{a_-}^{a_+} \frac{du}{f_n^2(u)} + 1 \right) \int_{a_-}^{a_+} \frac{du}{f_n^2(u)} .\end{aligned}\quad (159)$$

(again, note that $\beta_n^\alpha(z)$ are regular at $z = -2x_3$).

6.5 Explicit $U(1)$ solution

Now we present explicit formulae for the $U(1)$ monopole solution and study its properties.

Higgs/gauge fields

The Higgs field is given by:

$$\Phi = \sum_{n=0}^{\infty} \Phi_n(x_3) |n\rangle \langle n| , \quad (160)$$

it has axial symmetry, that is commutes with the number operator $c^\dagger c$. Explicitly:

$$\begin{aligned}\Phi_n &= \frac{\zeta_n}{\zeta_{n-1}} - \frac{\zeta_{n+1}}{\zeta_n} = \partial_3 \log \xi_n \\ &= (n-1)\eta_{n-2} - n\eta_{n-1}, \quad n > 0 \\ &= -\frac{\zeta_1}{\zeta_0} = -2x_3 - \frac{1}{\zeta_0}, \quad n = 0.\end{aligned}\quad (161)$$

To arrive at the third line we used the fact that

$$\frac{1}{\eta_n} - \frac{1}{\eta_{n+1}} = n\eta_{n-1} - (n+1)\eta_n ,$$

which follows immediately from the recursion relation for the ζ 's in (153). These fields are finite at $x_3 = 0$. Indeed as $x_3 \rightarrow 0$,

$$\Phi_n(x_3 = 0) = \sqrt{2} \left(\frac{\left(\frac{n-1}{2}\right)!}{\left(\frac{n-2}{2}\right)!} - \frac{\left(\frac{n}{2}\right)!}{\left(\frac{n-1}{2}\right)!} \right) . \quad (162)$$

At the origin:

$$\Phi_0(x_3 = 0) = -\sqrt{\frac{2}{\pi}}. \quad (163)$$

In the gauge where Φ is diagonal the component A_3 vanishes. In the same gauge the components A_1, A_2 (which we consider to be anti-hermitian) are given by:

$$\begin{aligned} A_c &= \frac{1}{2}(A_1 + iA_2), \quad A_{c^\dagger} = \frac{1}{2}(A_1 - iA_2) = -A_c^\dagger \\ A_c &= \xi^{-1}[\xi, c^\dagger] = c^\dagger \left(1 - \frac{\xi(n)}{\xi(n+1)}\right) \end{aligned} \quad (164)$$

Again we see that the matrix elements of A_c are all finite and non singular.

From (164) we deduce:

$$\begin{aligned} F_{12} &= 2 (\partial_c A_{c^\dagger} - \partial_{c^\dagger} A_c + [A_c, A_{c^\dagger}]) = \\ &= 2 \left(\left[\frac{\xi(n)}{\xi(n+1)} c, c^\dagger \frac{\xi(n)}{\xi(n+1)} \right] - 1 \right) = \\ &= 2 \sum_{n>0} \left(-1 + (n+1) \left(\frac{\xi(n)}{\xi(n+1)} \right)^2 - n \left(\frac{\xi(n-1)}{\xi(n)} \right)^2 \right) |n\rangle\langle n| + \\ &\quad + 2 \left(-1 + \left(\frac{\xi(0)}{\xi(1)} \right)^2 \right) |0\rangle\langle 0|, \end{aligned} \quad (165)$$

from which it follows, that:

$$\begin{aligned} B_3(n) &= 2 \left(1 - n \frac{\eta_{n-1}}{\eta_n} + (n-1) \frac{\eta_{n-2}}{\eta_{n-1}} \right) \\ B_c &= \frac{1}{2} (B_1 + iB_2) = c^\dagger \frac{\xi(n)}{\xi(n+1)} (\Phi(n) - \Phi(n+1)). \end{aligned} \quad (166)$$

with the understanding that at $n = 0$:

$$B_3(0) = 2 \left(1 - \frac{\zeta_1}{\zeta_0^2} \right). \quad (167)$$

Instantons, monopoles, and Yang ansatz

As in the ordinary gauge theory case the monopoles are the solutions of the instanton equations in four dimensions, that are invariant under translations in the fourth direction x_4 . We observe that the solution presented above (161),(164), can also be cast in the Yang form: Take $\xi = \xi(x_3, n)$ as in (164). Then $\partial_3 \xi$ commutes with ξ and we can write $\partial_3 \xi \xi^{-1} = \partial_3 \log \xi$. The formulae (66) yield exactly (164) and (161) with $\Phi = iA_4$.

All of the above and Toda lattice

At this point it is worth mentioning the relation of the noncommutative Bogomolny equations with the Polyakov's non-abelian Toda system on the semi-infinite one-dimensional lattice. Let us try to solve the equations (116) using the Yang ansatz and imposing the axial symmetry: we assume that $\xi(x_1, x_2, x_3) = \xi(n, x_3)$, $n = c^\dagger c$. Then the equation (67) for the x_4 -independent fields reduces to the system:

$$\partial_t(\partial_t g_n g_n^{-1}) - g_n g_{n+1}^{-1} + g_{n-1} g_n^{-1} = 0 \quad (168)$$

where

$$g_n(t) = \frac{e^{\frac{t^2}{2}}}{n!} \xi^2 \left(n, \frac{t}{2} \right) ,$$

(notice that $g_n(t)$ are ordinary matrices). In the $U(1)$ case we can write

$$g_n(t) = e^{\alpha_n(t)} ,$$

and rewrite (168) in a more familiar form:

$$\partial_t^2 \alpha_n + e^{\alpha_{n-1} - \alpha_n} - e^{\alpha_n - \alpha_{n+1}} = 0 \quad (169)$$

For $n = 0$ these equations also formally hold if we set $g_{-1} = 0$ (this boundary condition follows both from the Bogomolny equations and the same condition is imposed on the Toda variables on the lattice with the end-points).

Our Higgs field Φ_n has a simple relation to the α 's:

$$\Phi(x_3, n) = -2x_3 + \alpha'_n(2x_3) .$$

Our solution to (169) is:

$$\alpha_n = \frac{1}{2} t^2 + \log \left(\frac{n \zeta_{n-1}(t/2)}{\zeta_n(t/2)} \right) - \log(n!) . \quad (170)$$

It is amusing that Polyakov's motivation for studying the system (168) was the structure of loop equations for lattice gauge theory. Here we encountered these equations in the study of the *continuous*, but noncommutative, gauge theory, thus giving more evidence for their similarity.

We should note in passing that in the integrable non-abelian Toda system one usually has two 'times' t, \bar{t} , so that the equation (168) has actually the form²⁵:

$$\partial_t(\partial_{\bar{t}} g_n g_n^{-1}) - g_n g_{n+1}^{-1} + g_{n-1} g_n^{-1} = 0 . \quad (171)$$

It is obvious that these equations describe four-dimensional axial symmetric instantons on the noncommutative space with the coordinates t, \bar{t}, c, c^\dagger of which only half is noncommuting.

The mass of the monopole

In this section we restore our original units, so that 2θ has dimensions of (length)². From the formulae (156) we can derive the following estimates:

$$\Phi(n) \sim -\frac{1}{2r_n} = -\frac{1}{2\sqrt{x_3^2 + 2\theta n}} \quad n \neq 0, r \rightarrow \infty. \quad (172)$$

Instead, for $n = 0$ we have:

$$\begin{aligned} \Phi(0) &\sim -\frac{x_3}{\theta}, & x_3 \rightarrow +\infty \\ \Phi(0) &\sim -\frac{1}{2|x_3|}, & x_3 \rightarrow -\infty. \end{aligned} \quad (173)$$

The asymptotics of the magnetic field is clear from the Bogomolny equations and the behaviour of Φ . Thus, for example,

$$B_3(n) = -\partial_3 \Phi(n) = -\frac{x_3}{2r_n^3}, \quad n \neq 0, \quad (174)$$

and similarly for the other components of B . This is easily translated into ordinary position space, since, for large n , $B_i(n, x_3) \sim B_i(x_1^2 + x_2^2 \sim n, x_3)$. Therefore the magnetic field for large values of x_3 and n , or equivalently large x_i is that of a pointlike magnetic charge at the origin. However the $n = 0$ component of B_3 behaves differently for large positive x_3 :

$$B_3(n = 0) = -\partial_3 \Phi(0) = \frac{1}{\theta}. \quad (175)$$

Notice, that this is exactly the value of the B -field. Thus, in addition to the magnetic charge at the origin we have a flux tube, localized in a Gaussian packet in the (x_1, x_2) plane, of the size $\propto \theta$, along the positive x_3 axis. The monopole solution is indeed a smeared version of the Dirac monopole, wherein the Dirac string (the D-string!) is physical.

To calculate the energy of the monopole we use the Bogomolny equations to reduce the total energy to a boundary term:

$$\begin{aligned} \mathcal{E} &= \frac{1}{2g_{\text{YM}}^2} \int d^3x \left(\vec{B} \star \vec{B} + \vec{\nabla} \Phi \star \vec{\nabla} \Phi \right) = \\ &= \frac{1}{2g_{\text{YM}}^2} \int d^3x \left(\vec{B} + \vec{\nabla} \Phi \right)^2 - \frac{1}{2g_{\text{YM}}^2} \int d^3x \vec{\nabla} \cdot \left(\vec{B} \star \Phi + \Phi \star \vec{B} \right) = \\ &= \frac{2\pi\theta}{2g_{\text{YM}}^2} \int dx_3 \sum_n \langle n | \partial_3^2 \Phi^2 + 4\partial_c \left(\xi^2 (\partial_{c^\dagger} \Phi^2) \xi^{-2} \right) | n \rangle, \end{aligned} \quad (176)$$

where in the last line we switched back to the Fock space. Here we meet the

noncommutative boundary term, discussed in the section 2. Let us choose as the infrared regulator box the “region” where $|x_3| \leq L, 0 \leq n \leq N, L \sim \sqrt{2\theta N} \gg 1$. With the help of (10) the total integral in (176) reduces to the sum of two terms (up to the factor $\frac{\pi\theta}{g_{\text{YM}}^2}$):

$$4N \int_{-L}^L dx_3 \frac{\eta_{N-1}}{\eta_N} (\Phi_N^2 - \Phi_{N+1}^2) + \sum_{n=0}^N \partial_3 \Phi_n^2|_{x_3=-L}^{x_3=+L} . \quad (177)$$

The first line in (177) is easy to evaluate and it vanishes in $L \rightarrow \infty$ limit. The second line in (177) contains derivatives of the Higgs field evaluated at $x_3 = L \gg 0$ and at $x_3 = -L \ll 0$. The former is estimated using the $z \gg 0$ asymptotics in (156), and produces:

$$\sum_{n=0}^N \partial_3 \Phi_n^2(x_3 = L) \sim \frac{2\theta(N-1)}{L^3} + 2\frac{L}{\theta^2}$$

The diverging with L piece comes solely from the $n = 0$ term. Finally, the $x_3 = -L$ case is treated via $z \ll 0$ asymptotics in (157) yielding the estimate $\sim \theta N/L^3$ vanishing in the limit of large L, N .

Hence the total energy is given by

$$\mathcal{E} \propto \frac{2\pi\theta \times 2L}{2g_{\text{YM}}^2\theta^2} = \frac{2\pi L}{g_{\text{YM}}^2\theta} , \quad (178)$$

which is the mass of a string of length L whose tension is

$$T = \frac{2\pi}{g_{\text{YM}}^2\theta} . \quad (179)$$

Magnetic charge

It is instructive to see what is the magnetic charge of our solution. On the one hand, it is clearly zero:

$$Q \propto \int_{\partial(\text{space})} \vec{B} \cdot d\vec{S} = \int d^3x \vec{\nabla} \cdot \vec{B} = 0 \quad (180)$$

since the gauge field is everywhere non-singular. On the other hand, we were performing a θ -deformation of the Dirac monopole, which definitely had

magnetic charge. To see what has happened let us look at (180) more carefully. We again introduce the box and evaluate the boundary integral (180) as in (9):

$$\frac{Q}{2\pi} = \sum_{n=0}^N [B_3(x_3 = L, n) - B_3(x_3 = -L, n)] + 4N \int_{-L}^L dx_3 \frac{\eta_{N-1}}{\eta_N} (\Phi_N - \Phi_{N+1}) \quad (181)$$

It is easy to compute the sums

$$\begin{aligned} \sum_{n=0}^N B_3(x, n) &= \partial_3 \frac{\zeta_{N+1}}{\zeta_N} \\ 4N \int_{-L}^L dx_3 \frac{\eta_{N-1}}{\eta_N} (\Phi_N - \Phi_{N+1}) &= 4(N+1) \int \frac{\xi_N^2}{\xi_{N+1}^2} d \log \frac{\xi_N}{\xi_{N+1}} = \\ &= 2(N+1) \left(\frac{\xi_N}{\xi_{N+1}} \right)^2 \Big|_{x_3=-L}^{x_3=+L} = 2N \frac{\zeta_{N-1} \zeta_{N+1}}{\zeta_N^2} \Big|_{x_3=-L}^{x_3=+L}, \end{aligned} \quad (182)$$

and the total charge vanishes as:

$$Q = \left[2N \frac{\zeta_{N-1} \zeta_{N+1}}{\zeta_N^2} + \partial_3 \left(\frac{\zeta_{N+1}}{\zeta_N} \right) \right]_{x_3=-L}^{x_3=+L} \equiv 2(N+1)|_{x_3=+L} - 2(N+1)|_{x_3=-L}. \quad (183)$$

We can better understand the distribution of the magnetic field by looking separately at the fluxes through the “lids” $x_3 = \pm L$ of our box and through the “walls” $n = N$.

The walls contribute

$$\left[2N \frac{\zeta_{N-1} \zeta_{N+1}}{\zeta_N^2} \right]_{x_3=-L}^{x_3=+L} \sim -\frac{L}{\sqrt{L^2 + N}} \sim -1,$$

while the lids contribute $\sim +1$. Let us isolate the term $B_3(+L, n=0) \rightarrow +2$ (recall (173)). It contributes to the flux through the upper lid. The rest of the flux through the lids is therefore ~ -1 . Hence the flux through the rest of the “sphere at infinity” is -2 and it is roughly uniformly distributed (-1 contribute the walls and -1 the lids). So we get a picture of a spherical magnetic field of a monopole together with a flux tube pointing in one direction.

This spherical flux becomes observable in the naive $\theta \rightarrow 0$ limit, in which the string becomes localized at the point $x_3 = 0, n = 0$ (since the slope of the linearly growing $\Phi_0 \sim \frac{x_3}{\theta}$ becomes infinite). In the $\theta = 0$ limit we throw out this point and all of the string.

Thus we found an explicit analytic expression for a soliton in the U(1) gauge theory on a noncommutative space. The solution describes a magnetic monopole attached to a finite tension string, that runs off to infinity tranverse

to the noncommutative plane. This soliton has a clear reflection in type IIB string theory. If the gauge theory is realized as an $\alpha' \rightarrow 0$ limit of the theory on a D3-brane in the IIB string theory in the presence of a background NS B-field, then the monopole with the string attached is nothing but the D1-string ending on the D3-brane. What is unusual about the solution that we found is that it describes this string as a non-singular field configuration. Moreover, one can show that the long wavelength fluctuations of this string are described by the gauge theory^{40,41}.

6.6 Noncommutative $U(2)$ monopole.

Now we shall describe the noncommutative version of the 't Hooft-Polyakov monopole⁴⁹.

We are interested in the $U(2)$ gauge theory on the noncommutative three dimensional space. Recall that $H \approx \mathbf{C}^2$ is the Chan-Paton space, i.e. the fundamental representation for the commutative limit of the gauge group, and let e_0, e_1 denote an orthonormal basis in H . The noncommutative version of the fundamental representation is infinite dimensional, isomorphic to $H \otimes \mathcal{H}$. That is, the $U(2)$ matter fields Ψ belong to the space $\mathcal{H} \otimes Fun(x^3) \otimes (\mathcal{H} \otimes H)$, where the first two factors make it a representation of the algebra \mathcal{A}_θ of noncommutative functions on \mathbf{R}^3 , while the second two factors make it a representation of the $U(2)$ noncommutative gauge group. Actually, the latter is isomorphic to the group of (x^3 -dependent) unitary operators in the Hilbert space $\mathcal{H} \otimes H$. Now, the Hilbert space $\mathcal{H} \otimes H$ is isomorphic to \mathcal{H} itself:

$$|n\rangle \otimes e_\alpha \leftrightarrow |2n + \alpha\rangle . \quad (184)$$

Now the solutions can (and will) have a finite non-trivial magnetic BPS charge:

$$Q_m = \int dx^3 \text{Tr}_{\mathcal{H}} \partial_i (\text{Tr}_H \Phi B_i) , \quad (185)$$

where

$$B_i = \frac{\mathbf{i}}{2} \varepsilon_{ijk} [D_j, D_k] - \delta_{i3} \theta ,$$

and the Higgs field Φ approaches

$$\begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix} \otimes \mathbf{I}_{\mathcal{H}}$$

as $x_3^2 + 2\theta c^\dagger c \rightarrow \infty$.

In (145) we already found a two-component spinor vector-function

$$\Psi(z, \vec{x}) = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} ,$$

which obeys the equation (70). The solution to (70) is defined up to right multiplication by an element of $\text{Mat}_2(\mathcal{A}_\theta) \approx \mathcal{A}_\theta \otimes \text{End}(H)$: $\Psi \mapsto \Psi u$. Among these elements the unitary elements (i.e. the ones which solve the equation $uu^\dagger = u^\dagger u = 1$) are considered to be the gauge transformations. In the commutative setup one normalizes Ψ as follows:

$$\int dz \Psi^\dagger \Psi = \text{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \quad (186)$$

The normalization (186) only implies that the transformations u must obey: $u^\dagger u = 1$. The finite matrices u would then automatically obey $uu^\dagger = 1$. However, in the infinite-dimensional case this is not true. The operator uu^\dagger is merely a projection, which may have a kernel. The discussion on whether such projections should be viewed as gauge transformations in noncommutative gauge theory can be found in ^{30,31}.

Higgs/gauge field

Finally, given Ψ the solution for the gauge and Higgs fields is given explicitly by (124) where now one integrates over z from a_- to a_+ . We now are in position to calculate the components of the Higgs field and of the gauge field. We start with

$$\begin{aligned} \Phi &= \int dz z \Psi^\dagger \Psi = \\ &= \sum_{n \geq 0, \alpha, \gamma=0,1} \varphi_n^{\alpha\gamma} \cdot e_\alpha e_\gamma^\dagger \otimes |n-\alpha\rangle \langle n-\gamma| , \\ \text{where } \varphi_n^{\alpha\gamma} &= \int dz z \varepsilon_n^{\alpha,\dagger} \varepsilon_n^\gamma = -2x_3 \delta^{\alpha\gamma} + \int (b\beta_n^\alpha)(b+b^\dagger)(b\beta_n^\gamma) + n\beta_n^\alpha(b+b^\dagger)\beta_n^\gamma \\ &= -2x_3 \delta^{\alpha\gamma} + ((b\beta_n^\alpha)(b\beta_n^\gamma) - n\beta_n^\alpha\beta_n^\beta) \Big|_{a_-}^{a_+} . \end{aligned} \quad (187)$$

The component A_3 of the gauge field vanishes, just as in the case of the $U(1)$ solution of ³⁹:

$$A_3 = \int \Psi^\dagger \partial_3 \Psi = \int ((b\beta_n^\alpha) \partial_3 (b\beta_n^\gamma) + n\beta_n^\alpha \partial_3 \beta_n^\gamma) \cdot e_\alpha e_\gamma^\dagger \otimes |n-\alpha\rangle \langle n-\gamma| = \frac{1}{2} \partial_3 \int \Psi^\dagger \Psi = 0 . \quad (188)$$

The components A_1, A_2 can be read off the expression for the operator D :

$$\begin{aligned} D &= - \int dz \Psi^\dagger c^\dagger \Psi \\ &= \sum_{n \geq 0, \alpha, \gamma=0,1} D_n^{\alpha\gamma} \cdot e_\alpha e_\gamma^\dagger \otimes |n+1-\alpha\rangle \langle n-\gamma| , \\ \text{where } D_n^{\alpha\gamma} &= -\sqrt{n} (\beta_{n+1}^\alpha (b\beta_n^\gamma)) \Big|_{a_-}^{a_+} . \end{aligned} \quad (189)$$

The solution (187),(189) has several interesting length scales involved (recall that our units above are such that $2\theta = 1$):

$$\theta|a_+ - a_-|, \quad \sqrt{\theta}, \quad \frac{1}{|a_+ - a_-|}.$$

By shifting x_3 we can always assume that $a_- = 0, a_+ = a > 0$.

Suspended D-string

In this section we set θ back to $\frac{1}{2}$. As discussed in⁴¹ the spectrum of the operators D_A , $A = 1, 2, 3, 4$ determines the “shape” of the collection of D-branes the solution of the generalized IKKT model¹⁵ corresponds to. To “see” the spatial structure of our solution let us concentrate on the $\langle 0|\Phi|0\rangle$ piece of the Higgs field, for it describes the profile of the D-branes at the core of the soliton. From (187) we see that

$$\langle 0|\Phi|0\rangle = \begin{pmatrix} \rho_+ & 0 \\ 0 & \rho_- \end{pmatrix},$$

where $\rho_+ = \varphi_0^{00}, \rho_- = \varphi_1^{11}$.

Let us look specifically at the component ρ_+ of the Higgs field:

$$\begin{aligned} \rho_+ &= -\frac{1}{2} \frac{\partial}{\partial x_3} \log \left(\int_0^a dp e^{-2x_3 p - \frac{1}{2} p^2} \right), \\ &= -2x_3 + \langle \langle p \rangle \rangle_{2x_3}^{a+2x_3}, \\ &= -2x_3 - 2 \frac{e^{-\frac{(a+2x_3)^2}{4}} - e^{-\frac{(2x_3)^2}{4}}}{\gamma(a+2x_3) - \gamma(2x_3)}, \end{aligned} \quad (190)$$

where

$$\langle \langle \mathcal{O} \rangle \rangle_\alpha^\beta = \frac{\int_\alpha^\beta \mathcal{O} e^{-\frac{p^2}{4}} dp}{\int_\alpha^\beta e^{-\frac{p^2}{4}} dp}, \quad \gamma(z) = \int_0^{\frac{z}{2}} dp e^{-\frac{p^2}{4}}. \quad (191)$$

The $\langle \langle \dots \rangle \rangle$ representation of the answer helps to analyze the qualitative behavior of the profile of ρ_+ . Clearly, the truncated Gaussian distribution which enters the expectation values $\langle \langle \dots \rangle \rangle$ in (190) favors $p \approx 0$ if $\alpha < 0 < \beta$, $p \approx \alpha$ for $\alpha > 0$ and $p \approx \beta$ for $\beta < 0$. Thus,

$$\begin{aligned} \rho_+ &\sim 0, & x_3 &> 0 \\ \rho_+ &\sim -2x_3, & 0 > x_3 &> -\frac{1}{2}a \\ \rho_+ &\sim a, & -\frac{1}{2}a &> x_3. \end{aligned} \quad (192)$$

This behavior agrees with the expectations about the tilted D1-string suspended between two D3-branes separated by a distance $|a|$. The eigenvalue ρ_+ corresponds roughly to the the transverse coordinate of the D1 string, that runs from a at large negative x_3 to 0 at large positive x_3 . In between the linear behavior of the Higgs field corresponds to the D1 string tilted at the critical angle. Indeed, for large $a \gg 1$, in the region $0 > x_3 > -\frac{1}{2}a$ this solution looks very similar to that of a single fluxon⁴⁰. Another eigenvalue of $\langle 0|\Phi|0\rangle$, ρ_- , is given by:

$$\rho_- = \frac{2x_3(2x_3+a)M+M^2-(2x_3+a)^2-e^{-2x_3a-\frac{a^2}{2}}}{M(2x_3+a-2x_3M)} \quad (193)$$

where $M = e^{-2x_3a-\frac{a^2}{2}} + (2x_3+a) \int_0^a e^{-2x_3p-\frac{p^2}{2}} dp$

At this point, however, we should warn the reader that only the eigenvalues of the full, $2\infty \times 2\infty$ operator Φ should be identified with the D-brane profile. The components ρ_{\pm} do not actually coincide with any of them. The eigenvalues of Φ , as it follows from the representation (161), are located between 0 and a , which is also what we expect from the dual D-brane picture⁴⁵.

6.7 Tension of the monopole string versus that of D-string

In this section we shall match the tension of the string we observed in the $U(1)$ monopole solution to that of D-string ending on the D3-brane in the presence of the constant B -field. As already mentioned, a D-string ending on a D3-brane in the presence of the constant B -field bends. To analyze this bending one could use the exact solution of the Dirac-Born-Infeld theory⁶³, the B -deformed spike solutions of¹². However, for our qualitative analysis, it is sufficient to look at the linearized equations. If we replace the DBI Lagrangian by its Maxwell approximation, then the BPS equations in the presence of the B -field will have the form:

$$B_{ij} + F_{ij} + \sqrt{\det g} \varepsilon_{ijk} g^{kl} \partial_l \Phi = 0, \quad (194)$$

where we should use the closed string metric (46). The solution of (194) is:

$$\Phi = B \left(1 + \left(\frac{\theta}{2\pi\alpha'} \right)^2 \right) x_3 - \frac{1}{2r}, \quad r^2 = x_3^2 + \frac{1}{\left(1 + \left(\frac{\theta}{2\pi\alpha'} \right)^2 \right)} (x_1^2 + x_2^2). \quad (195)$$

The linearly growing piece in Φ should be interpreted as a global rotation of the D3-brane, by an angle ψ , $\tan\psi = \frac{\theta}{(2\pi\alpha')}$. This conclusion remains correct even after the full non-linear BPS equation is solved (see⁶³. Notice however

that we fix $G_{ij} = \delta_{ij}$ instead of $g_{ij} = \delta_{ij}$ as in ⁶³). The singular part of Φ , the spike, represents the D-string. If we rotate the brane, then the spike forms an angle $\frac{\pi}{2} - \psi$ with the brane. If we project this spike on the brane, then the energy, carried by its shadow per unit length, is related to the tension of the D-string via:

$$\frac{T_{D1}}{\sin\psi} = \frac{1}{2\pi\alpha'g_s} \frac{\sqrt{(2\pi\alpha')^2 + \theta^2}}{\theta} = \frac{(2\pi\alpha')^2 + \theta^2}{2\pi g_{\text{YM}}^2 (\alpha')^2 \theta}. \quad (196)$$

However, this is not the full story. The endpoint of the D-string is a magnetic charge, which experiences a constant force, induced by the background magnetic field. If we had introduced a box $-L \leq x_3 \leq x_3$ of the extent $2L$ in the x_3 -direction, then in order to bring a tilted D-string into our system from outside $x_3 > L$ of the box to $x_3 = 0$ we would have had to spend an energy equal to $\frac{T_{D1}}{\sin\psi} L$, but we would have been helped by the magnetic force, which would decrease the work done by⁴⁵

$$\frac{2\pi L}{g_{\text{YM}}^2} B_3 = \frac{2\pi L}{g_{\text{YM}}^2} B_{12} g^{11} g^{22} \sqrt{g} \sim \frac{1}{(2\pi\alpha'g_{\text{YM}})^2 \theta}.$$

In sum, the energy of the semi-infinite D-string in the box per unit length in the x_3 direction will be given by

$$\frac{2\pi}{g_{\text{YM}}^2 \theta}. \quad (197)$$

This expression coincides with our tension (179). On dimensional grounds, non-commutative gauge theory cannot produce any other dependence of the tension on θ but that given in (179).

7 Conclusions and historic remarks

To conclude these lectures I would start with a very short survey of the topics not included into them. The abovementioned factorization of the OPE algebra of the open string vertex operators was argued to be useful in analyzing various instability issues, condensation of tachyons, responsible for the decays of the unstable D-branes^{83,7}. Moreover, noncommutative gauge theories allow to see both (many of) the non-BPS D-branes as classical solutions and the processes of their decays^{1,42,41}.

My interest in noncommutative theories was prompted by the work with V. Fock, A. Rosly and K. Selivanov in 1991 on the geometric quantization

of higher-dimensional Chern-Simons theories²⁸, where we encountered moduli spaces of codimension two four dimensional foliations with flat $U(1)$ connections on the fibers as the classical phase space of the five dimensional theory (in modern terms these would be supersymmetric cycles, if they were holomorphic). Previously these theories were encountered in the context of gauge anomalies in²⁶. An example of such foliation would be an irrational foliation of a four-torus, the space suited for the analysis by noncommutative geometry. In 1994 with G. Moore and S. Shatashvili I tried to understand S-duality of the $\mathcal{N} = 4$ gauge theory on ALE manifolds following⁹² and then together with A. Losev we came from that to the attempt of constructing the four dimensional analogue of the two dimensional RCFT⁵⁹. In the course of this study we realized that the construction of⁶⁷ of the instanton moduli spaces on ALE manifolds, whose Euler characteristics used in⁹² were generated by modular forms actually gave nontrivial answers already in the $U(1)$ case, which is almost impossible to achieve by the ordinary gauge fields. With the help of I. Grojnowski we realized that Nakajima constructed not the instantons but the torsion free sheaves, who had no gauge theory interpretation (but were used by algebraic geometers to construct compactifications of instanton moduli spaces, e.g. Gieseker compactification). For some time these moduli spaces were a mystery. This mystery was getting deeper after discovery of D-branes by J. Polchinski⁷⁶ and the realization by E. Witten and M. Douglas⁹⁵ that the $D(p-4)$ -branes within Dp -brane are instantons. Now, the $D4$ -brane carries only a $U(1)$ gauge field on it, so what is the $D0$ -brane which is dissolved inside? This question is hard to ask if the gauge theory is not decoupled from the rest of the ten dimensional string theory, but it was soon realized that a constant B -field may help (in the similar M-theory context turning on C -field helped to decouple fivebrane from the eleven dimensional sugra modes³). Things came together after we had a very fruitful lunch with A. Schwarz in ITP, Santa Barbara in the spring of 1998, where he explained to me his paper with A. Connes and M. Douglas, and I explained to him what we have learned about torsion free sheaves with other authors of⁵⁹. After brief discussion we became confident that the gauge fields on the noncommutative \mathbf{R}^4 must be *i*) constructed with the help of ADHM construction applied to the deformed ADHM equations⁶⁵, which was shown by Nakajima to parameterize torsion free sheaves⁶⁸ on \mathbf{C}^2 ; *ii*) obey instanton equations on the noncommutative space. It was a matter of simple algebraic manipulations to check that this was the case⁷⁴. But then the devil of doubts started to hunt me. Several people asked us whether the $U(1)$ instantons were non-singular. The relation to large N gauge theory suggested they didn't exist, for the instanton effects usually die out in the large N limit. Explicit computations^{11,29} seemed to imply that the $U(1)$ instantons had to

live over the space of complicated topology. Also, with D. Gross we were finding solutions to the BPS equations in three dimensions which seemed to follow from ADHM ansatz and yet did not solve Bogomolny equation everywhere in the space. Later on we realized that these solutions were obtained from the true solution by applying the “generalized” gauge transformation u , discussed below the equation(186), and which actually could produce a source. But in the spring of 2000 I was not yet aware of that and during the talk at CIT-USC center I was forced by E. Witten to announce that noncommutative instantons existed not on noncommutative \mathbf{R}^4 but on some space, obtained from \mathbf{R}^4 by a sequence of blowups, whose commutative limit was described in¹¹. This point of view was immediately criticized by E. Witten himself, for it was not consistent with many results on the counting of states of D-branes. Then a week later D. Gross and I found true sourceless solution to the monopole equations³⁹. And later on I realized that the explicit formulae for the $U(1)$ instanton gauge field presented in⁷⁴ were harmed by the same plague: they were written in the singular gauge. Almost at the same time this conclusion was reached by K. Furuuchi^{31,72}.

Independently of all this story, it is natural to look for the mechanism of the confinement in the noncommutative gauge theory, hoping to transport these results to the large N ordinary gauge theory. We find⁴¹ all sorts of magnetically charged objects in the gauge theory, whose mass/tension goes to zero in the limit $\theta \rightarrow \infty$. Whether this could provide the sought for mechanism for the confinement via condensation of magnetic charges still remains to be seen.

As far as other extensions of the work reported above are concerned let us mention a few. First of all, it is quite interesting to construct noncommutative instantons on other spaces, for example ALE manifolds^{74,58}, tori or K3 manifolds³³, or Del Pezzo surfaces or cotangent bundles to curves (see⁶⁹). In the latter three cases the very notion of the underlying noncommutative manifold is missing (for orbifold K3’s one can make orbifolds of noncommutative tori, though³). One could also try to construct supersymmetric gauge field configurations in higher dimensions (see⁷⁰ for some relevant algebraic geometric results). Also, $SO(n)$ and $Sp(n)$ theories¹⁰ are quite curious to look at. The construction of the ordinary instantons⁹⁵ in these theories with D-brane techniques shows many interesting surprises²². It is also quite interesting to generalize the noncommutative twistor approach of⁵² to more general spaces. Another subject is the usage of the gauge theory/string duality in the form of supergravity duals of noncommutative gauge theories⁴⁶. One can construct many sugra duals of the solitons in noncommutative theories and study their strong coupling behaviour⁷⁹. Yet another interesting topic is the construction

of electrically charged solitons. They are important in the realizing of closed strings within open string field theories in the lines of⁸⁷.

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